Limits to clock synchronization induced by completely dephasing communication channels

V. Giovannetti,[1,*] S. Lloyd,[2,†] L. Maccone,[1,‡] and M. S. Shahriar[1,§]
[1]*Research Laboratory of Electronics, Massachusetts Institute of Technology, 50 Vassar Street, Cambridge, Massachusetts 02139*
[2]*Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*
(Received 26 October 2001; published 17 June 2002)

Clock synchronization procedures are analyzed in the presence of imperfect communications. In this context we show that there are physical limitations, which prevent one from synchronizing distant clocks when the intervening medium is completely dephasing, as in the case of a rapidly varying dispersive medium.

DOI: 10.1103/PhysRevA.65.062319 PACS number(s): 03.67.Hk, 06.30.Ft, 03.65.Ta

INTRODUCTION

There are two main kinds of protocols for achieving clock synchronization. The first is the ''Einstein synchronization protocol'' [1] in which a signal is sent back and forth between one of the clocks (say Alice's clock) and the other clocks. By knowing the signal speed dependence on the intermediate environment, it is possible to synchronize all the clocks with Alice's. The other main protocol is the ''Eddington slow clock transfer'' [2]: after locally synchronizing it with hers, Alice sends a clock (i.e., a physical system that evolves in time with a known time dependence) to all the other parties. The clock's transfer must of course be perfectly controllable, as one must be able to predict how the clock will react to the physical conditions encountered *en route*, which may shift its time evolution. Moreover, since any acceleration of the transferred clocks introduces a delay because of relativistic effects, one must suppose that the transfer is performed ''adiabatically slowly,'' i.e., such that all accelerations are negligible. Notice that the above protocols can be implemented using only classical resources: peculiar quantum features such as entanglement, squeezing, etc., are not needed. In what follows, such synchronization schemes will be referred to as ''classical protocols.''

A recently proposed quantum clock synchronization protocol [3] was found [4] to be equivalent to the Eddington slow clock synchronization. The application of entanglement purification to improve quantum clock synchronization in the presence of dephasing was attempted without success in [5]. One might think there were other ways to implement a synchronization scheme that employs quantum features such as entanglement and squeezing, but this paper shows that this is not the case. In fact, it will be shown that quantum mechanics does not allow one to synchronize clocks if it would not be possible to also employ one of the classical protocols, which one can always employ if the channel is perfect or if its characteristics are controllable. However, the relevance of quantum mechanics to the clock synchronization procedures should not be underestimated, since there exist schemes that exploit quantum mechanics to achieve a (classically not allowed) increase in the accuracy of classical clock synchronization protocols, such as the one obtainable exploiting entangled systems [6–8].

The presented discussion also takes into account the possibility that the two distant parties who want to synchronize their clocks (say Alice and Bob) and who are localized in space can entangle their systems by exchanging a certain number of quantum states, and the possibility that they may employ the ''wave function collapse'' [3], through postselection measurements. The intuitive idea behind the proof is as follows. To synchronize clocks, Alice and Bob must exchange physical systems such as clocks or pulses of light that include timing information. But any effect, such as rapidly varying dispersion, that randomizes the relative phases between energy eigenstates of such systems completely destroys the timing information. Any residual information, such as entanglement between states with the same energy, cannot be used to synchronize clocks as shown below.

The paper is organized as follows. In Sec. I the analytic framework is established. In Sec. II the clock synchronization procedure is defined and the main result is derived. In particular, in Sec. II A the exchange of quantum information between Alice and Bob is analyzed and in Sec. II B the analysis is extended to include partial measurements and postselection schemes in the synchronization process.

I. THE SYSTEM

Assume the following hypotheses that describe the most general situation in which two distant parties communicate through a noisy environment:

(1) Alice and Bob are *separate* entities that initially are *disjoint*. They belong to the same inertial reference frame and communicate by exchanging some physical system.

(2) The environment randomizes the phases between different energy eigenstates of the exchanged system while in transit.

From these hypotheses it will be shown that Alice and Bob cannot synchronize their clocks.

In Sec. I A we explain the first hypothesis by giving its formal consequences. In Sec. I B we analyze the second hypothesis and explain how it describes a dephasing channel.

A. First hypothesis

The first hypothesis states the problem and ensures that initially Alice and Bob do not already share any kind of

*Email address: vittorio@mit.edu
†Email address: slloyd@mit.edu
‡Email address: maccone@mit.edu
§Email address: smshahri@mit.edu

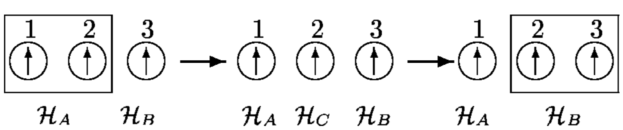
system that acts as a synchronized clock. By *separate* we mean that at any given time Alice and Bob cannot gain access to the same degrees of freedom and there is no direct interaction between Alice's and Bob's systems. This can be described by the following properties of the system's Hilbert space and Hamiltonian. At time t the Hilbert space of the global system can be written as

$$\mathcal{H}=\mathcal{H}_A(t)\otimes\mathcal{H}_C(t)\otimes\mathcal{H}_B(t), \tag{1}$$

where the Hilbert space $\mathcal{H}_A(t)$ refers to the system on which Alice can operate at time t, $\mathcal{H}_B(t)$ refers to Bob's system, and $\mathcal{H}_C(t)$ describes the systems on which neither of them can operate. The time dependence in Eq. (1) does not imply that the global Hilbert space changes in time, but it refers to the possibility that a system that was previously under Alice's influence has been transferred to Bob (or vice versa), after a transient time at which it cannot be accessed by any of them. Since information *must* be encoded into a physical system, this mechanism describes any possible communication between them. Moreover, the Hamiltonian of the system can be written as

$$H(t)=H_A(t)+H_B(t)+H_C(t), \tag{2}$$

where the time dependent $H_A(t)$ and $H_B(t)$ evolve the states in $\mathcal{H}_A$ and $\mathcal{H}_B$ under the control of Alice and Bob, respectively, while $H_C(t)$ evolves the system in transit between them when it is not accessible. As a consequence of Eq. (1), at time t the three terms on the right side of Eq. (2) commute, since they act on different Hilbert spaces. For the same reason any operator under the influence of Alice at time t commutes with all Bob's operators at the same time. A simple example may help explain this formalism. Consider the situation in which the system is composed of three 1/2 spin particles (qubits). A possible communication is then modeled by the sequence



i.e., initially Alice's Hilbert space $\mathcal{H}_A$ contains spins 1 and 2, and Bob owns only spin 3. Alice then encodes some information on spin 2 (eventually entangling it with spin 1), and sends it to Bob. There will be a time interval in which none of them can access spin 2, and this situation corresponds to having spin 2 belonging to $\mathcal{H}_C$. Finally, Bob receives spin 2, and his Hilbert space $\mathcal{H}_B$ describes both spins 2 and 3. Notice that the form of the Hamiltonian in Eq. (2), where no interaction terms are present, allows each of them to act, at a given time t, only on the spins that live in their own Hilbert space at time t. An analogous description applies also to more complicated scenarios, such as the exchange of light pulses. In this case, causality constraints allow Alice and Bob to act only on localized traveling wave modes of the electromagnetic field. Thus, also here, it is possible to define a traveling system Hilbert space $\mathcal{H}_C$ that factorizes as in Eq. (1). From the above example, it is easy to see that in each communication exchange it is possible to define a *departure* time t_s after which the sender cannot act anymore on the system in transit, and an *arrival* time t_r before which the receiver cannot yet act on such system. It is between these two times that the exchanged system belongs to $\mathcal{H}_C$.

In hypothesis 1 by initially *disjoint* we mean that Alice and Bob do not share any information prior to the first communication exchange. In particular this means that, before they start to interact, the state of the system factorizes as

$$|\Psi\rangle=|\phi\rangle_A\otimes|\varphi\rangle_B, \tag{3}$$

i.e., the initial state is not entangled and they do not share any quantum information. Here $|\phi\rangle_A$ is the state of Alice's system evaluated at the time at which she starts to act, while $|\varphi\rangle_B$ is the state of Bob's system evaluated at the time at which *he* starts to act. For ease of notation, the tensor product symbol $\otimes$ will be omitted in the following except when its explicit presence helps comprehension.

B. Second hypothesis

The second hypothesis imposes limitations to the information retrieved from the exchanged signal. The dephasing of the energy eigenstates describes the nondissipative noise present in most nonideal communication channels and implies a certain degree of decoherence in any quantum communication between Alice and Bob. Define $|e,d\rangle$ as the eigenstate relative to the eigenvalue $\hbar\omega_e$ of the free Hamiltonian of the exchanged system C. The label d takes into account possible degeneracy of such eigenstate. We assume that during the travel, when neither Alice nor Bob can control the exchanged system in $\mathcal{H}_C$, the states $|e,d\rangle$ undergo the transformation

$$|e,d\rangle\rightarrow e^{-i\varphi_e}|e,d\rangle, \tag{4}$$

where the random phase $\varphi_e\in[0,2\pi]$ is independent of d. The channel dephasing arises when different energy eigenstates are affected by different phase factors φ_e. For this reason the dephasing is characterized by the joint probability function $p_\epsilon(\varphi_e,\varphi_{e'})$ that weights the probability that the energy levels $|e,d\rangle$ and $|e',d\rangle$ are affected by the phases φ_e and $\varphi_{e'}$, respectively. The parameter $\epsilon\in[0,1]$ measures the degree of decoherence in the channel. In particular, $\epsilon=1$ describes the case of complete decoherence, where the phases relative to different energy eigenstates are completely uncorrelated, namely $p_\epsilon(\varphi_e,\varphi_{e'})$ is a constant. On the other hand, $\epsilon=0$ describes the case of no decoherence, where each energy eigenstate acquires the same phase, namely $p_\epsilon(\varphi_e,\varphi_{e'})\rightarrow\delta(\varphi_e-\varphi_{e'})/2\pi$. Written in the energy representation, the channel density matrix ϱ_c evolves, using Eq. (4), as

$$\varrho_c=\sum_{ee'}P_e\varrho_cP_{e'}\rightarrow\sum_{ee'}e^{-i(\varphi_e-\varphi_{e'})}P_e\varrho_cP_{e'}, \tag{5}$$

where $P_e=\Sigma_d|e,d\rangle\langle e,d|$ is the projection operator on the channel eigenspace of energy $\hbar\omega_e$. Taking into account the stochasticity of the evolution (4), the right-hand term of Eq. (5) must be weighted by the probability distribution $p_\epsilon(\varphi_e,\varphi_{e'})$, resulting in

$$\varrho_c \rightarrow \sum_{ee'} \delta^{(\epsilon)}_{ee'} P_e \varrho_c P_{e'}, \tag{6}$$

where

$$\delta^{(\epsilon)}_{ee'} = \int_0^{2\pi} d\varphi_e \int_0^{2\pi} d\varphi_{e'} p_\epsilon(\varphi_e,\varphi_{e'}) e^{-i(\varphi_e-\varphi_{e'})}. \tag{7}$$

The width of the function $\delta^{(\epsilon)}_{ee'}$ decreases with ϵ, so that $\delta^{(\epsilon=0)}_{ee'}$ is independent of e and e' and the state is unchanged, while $\delta^{(\epsilon=1)}_{ee'}$ is the Kronecker δ and the state suffers from decoherence in the energy eigenstate basis.

The dephasing process of Eq. (4) can be derived assuming a time dependent Hamiltonian $H_C(t)=H^o_C+H'_C(t)$, where H^o_C is the free evolution of the system with eigenstates $|e,d\rangle$ and $H'_C(t)$ is a stochastic contribution that acts on the system in a small time interval δt by shifting its energy eigenvalues by a random amount ν_e, such that $\nu_e\delta t=\varphi_e$. In fact, in the limit $\delta t\rightarrow 0$, the evolution of the exchanged system is described by

$$U_C(t_r,t_s)=\exp\left\{-i\sum_e P_e[\omega_e(t_r-t_s)+\varphi_e]\right\}, \tag{8}$$

where $\hbar\omega_e$ is the energy eigenvalue of the exchanged system relative to the eigenvector $|e,d\rangle$ and t_r and t_s are the exchanged system's arrival and departure times, respectively, introduced in Sec. I A [9]. Notice that for φ_e independent of e (which corresponds to the case $\epsilon=0$), U_C reduces to the deterministic free evolution operator $\exp[-(i/\hbar)H^o_C(t_r-t_s)]$, apart from an overall phase term.

It might be interesting to consider the simpler case in which the random phase φ_e can be written as $\omega_e\theta$ with the random term θ independent of e. In this case, Eq. (8) simplifies to

$$U_C(t_r,t_s)=\exp\left[-\frac{i}{\hbar}H^o_C(t_r-t_s+\theta)\right]. \tag{9}$$

This last situation depicts the case in which all signals exchanged between Alice and Bob are delayed by an amount θ. As an example consider light signals that encounter a medium with unknown (possibly varying) refractive index or a traveling ''clock'' that acquires an unpredictable delay. The situation described by Eq. (8) is even worse, since not only may such a delay be present, but also the wave function of the system is degraded by dispersion effects. In both cases, the information on the transit time t_r-t_s that may be extracted from $U_C(t_r,t_s)$ depends on the degree of randomness of φ_e. In particular, if φ_e is a completely random quantity (i.e., for $\epsilon=1$), no information on the transit time can be obtained.

This, of course, prevents the possibility of using classical synchronization protocols, where unknown delays in either the signal travel time or in the exchanged clock prove to be fatal. One might think that by exploiting the apparently nonlocal properties of quantum mechanics (e.g., entanglement), these limits can be overcome. In the following sections we will show that this is not the case.

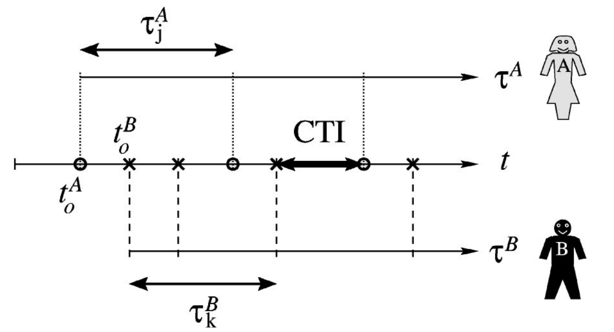


FIG. 1. Comparison between the times τ^A and τ^B of Alice and Bob's clocks. The center line represents the ''absolute'' time as measured by an external clock. The small circles represent the times of events that take place on Alice's side, while the crosses represent those on Bob's side. The upper line is the time as measured by Alice's clock: she only has direct access to the proper time intervals such as τ^A_j. Analogously, the lower line represents Bob's proper time. To achieve clock synchronization, Alice and Bob need to recover a connecting time interval (CTI) such as the one shown.

II. CLOCK SYNCHRONIZATION

In this section we analyze the clock synchronization schemes in detail and show the effect of a dephasing communication channel.

How does synchronization take place? Define t^A_0 and t^B_0 as the initial times of Alice and Bob's clocks as measured by an external clock. (Of course, since they do not have a synchronized clock to start with, they cannot measure t^A_0 and t^B_0.) Alice and Bob will be able to synchronize their clocks *if and only if* they can recover the quantity $t^A_0-t^B_0$, or any other time interval that connects two events that happen one on Alice's side and the other on Bob's side. Each of them has access to the times at which events on her/his side happen and can measure such events only relative to their own clocks. We will refer to these quantities as ''proper time intervals'' (PTIs). For Alice such quantities are defined as $\tau^A_j=t^A_j-t^A_0$, where t^A_j is the time at which the jth event took place as measured by the external clock. Analogously for Bob we define his PTI as $\tau^B_k=t^B_k-t^B_0$. If Alice and Bob share the data regarding their own PTIs, they cannot achieve synchronization: they need also a ''connecting time interval'' (CTI), i.e., a time interval that connects an event that took place on Alice's side with an event that took place on Bob's side as shown in Fig. 1.

Within this framework, consider the case of Einstein's and Eddington's clock synchronizations. In the Einstein clock synchronization the PTIs on Alice's side are the two times at which she sent and received back the signal she sends Bob. Bob's PTI is the time at which he bounces back the signal to Alice. The CTI in this case measures the time difference between the events "Alice sends the signal" and "Bob bounces the signal back." The protocol allows Alice to recover the CTI by simply dividing by two the time difference between her two PTIs. The analysis of Eddington's slow clock transfer is even simpler. In this case Bob's PTI is the time at which Bob looks at the clock Alice has sent him after synchronizing it with hers. The CTI is, for example, the time difference between the event "Bob looks at the clock sent by Alice" and "on Alice's side it is noon": Bob can recover it just looking at the time shown on the clock he received from Alice.

In this paper we show that in the presence of a dephasing communication channel (as described in hypothesis 2), there is no way in which Alice and Bob may achieve a CTI. The best that they can do is to collect a series of PTIs related to different events and a collection of CTI transit times corrupted by the noisy communication line: clock synchronization is thus impossible.

A. Timing information exchange

In this section we analyze the exchange of quantum information between Alice and Bob in the presence of dephasing.

Starting from the state $|\Psi\rangle$ of Eq. (3), Alice's and Bob begin to act on their systems at two times (that are not necessarily the same), in order to get ready for the information transfer. Without loss of generality one can assume that these two times coincide with their own time origins, i.e., t_0^A and t_0^B. This means that, at those two times, they introduce time dependent terms in the system Hamiltonian

$$\begin{aligned} H_A^o &\to H_A(t) \equiv H_A^o + H_A'(t-t_0^A), \\ H_B^o &\to H_B(t) \equiv H_B^o + H_B'(t-t_0^B), \end{aligned} \tag{10}$$

where H_A^o and H_B^o are the free Hamiltonians of Alice's and Bob's systems and $H_A'(t-t_0^A)$ and $H_B'(t-t_0^B)$ characterize the most general unitary transformations that they can apply to their systems. These last terms are null for $t<t_0^A$ and $t<t_0^B$ (when they have not yet started to act on their systems). Notice that according to Eq. (1), also the domains of $H_A(t)$ and $H_B(t)$ may depend on time.

Suppose first that Alice is going to send a signal to Bob. Define t_s^A the departure time at which Alice sends a message to Bob encoding it on a system described by the Hilbert space $\mathcal{H}_c$. This implies that the system she has access to will be $\mathcal{H}_a$ up to t_s^A and $\mathcal{H}_{a'}$ afterward, so that $\mathcal{H}_a = \mathcal{H}_{a'} \otimes \mathcal{H}_c$. In the same way, defining t_r^B as the arrival time on Bob's side, we may introduce a space $\mathcal{H}_{b'} = \mathcal{H}_b \otimes \mathcal{H}_c$ that describes the Hilbert space on which Bob acts after t_r^B. The label A on t_s^A refers to the fact that the event of sending the message happens locally on Alice's side, so in principle she can measure such a quantity as referred to her clock as the PTI $\tau_s^A = t_s^A - t_0^A$. Analogous consideration applies to Bob's receiving time t_r^B and Bob's PTI $\tau_r^B = t_r^B - t_0^B$.

Consider the situation of Fig. 2 in which, for explanatory purposes, $t_0^A < t_0^B < t_s^A < t_r^B$. Start from the group property of the time evolution operators

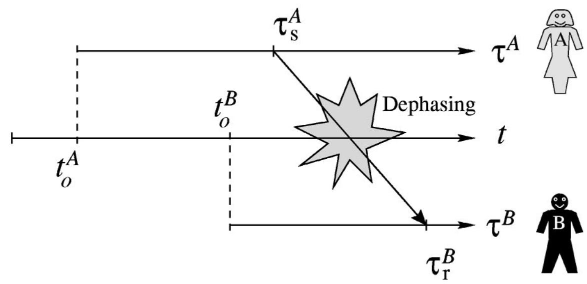


FIG. 2. Alice sends Bob a message encoded into a quantum system C at time t_s^A (her proper time τ_s^A) and Bob receives it at time t_r^B (his proper time τ_r^B). During the travel the system C undergoes dephasing.

$$U(t,0) = U(t,t')U(t',0), \tag{11}$$

and the commutativity of the operators that act on the distinct spaces of Alice and Bob. It is easy to show that for $t_s^A \leqslant t \leqslant t_r^B$ the state of the system is given by

$$|\Psi(t)\rangle = U_b(t,t_0^B)U_{a'}(t,t_s^A)U_c(t,t_s^A)U_a(t_s^A,t_0^A)|\Psi\rangle, \tag{12}$$

where $U_x(t,t')$ is the evolution operator in space $\mathcal{H}_x$ and

$$|\Psi\rangle \equiv U_a(t_0^A,0)U_b(t_0^B,0)|\Psi(0)\rangle \tag{13}$$

is the initial state as far as Alice and Bob are concerned, defined in Eq. (3). By hypothesis 1 this state does not contain any usable information on t_0^A and t_0^B. In Eq. (12) notice that up to time t_s^A the systems $\mathcal{H}_c$ and $\mathcal{H}_{a'}$ are evolved together by U_a. Analogously, for $t \geqslant t_r^B$ after Bob has received the system Alice sent him, one has

$$|\Psi(t)\rangle = U_{a'}(t,t_r^B)U_{b'}(t,t_r^B)|\Psi(t_r^B)\rangle. \tag{14}$$

Joining Eqs. (12) and (14), it follows

$$\begin{aligned} |\Psi(t)\rangle = {} & U_{b'}(t,t_r^B)U_b(t_r^B,t_0^B)U_{a'}(t,t_s^A)U_c(t_r^B,t_s^A) \\ & \times U_a(t_s^A,t_0^A)|\Psi\rangle. \end{aligned} \tag{15}$$

The time dependence of Alice's and Bob's Hamiltonians (10) allows to write their unitary evolution operators as functions of their PTIs, i.e.,

$$\begin{aligned} U_\alpha(t',t'') &= \overleftarrow{\exp}\left[-\frac{i}{\hbar}\int_{t''}^{t'} dt [H_\alpha^o + H_\alpha'(t-t_0^A)]\right] \\ &= \overleftarrow{\exp}\left[-\frac{i}{\hbar}\int_{t''-t_0^A}^{t'-t_0^A} dt [H_\alpha^o + H_\alpha'(t)]\right] \\ &\equiv \overline{U}_\alpha(\tau'^A,\tau''^A), \end{aligned} \tag{16}$$

where $\alpha = a, a'$ and the arrow indicates time ordering in the expansion of the exponential. Analogously

$$U_\beta(t',t'') \equiv \overline{U}_\beta(\tau'^B,\tau''^B), \tag{17}$$

with $\beta = b, b'$. Now Eq. (15) can be rewritten as

$$|\Psi(t)\rangle = \overline{U}_{b'}(\tau^B,\tau^B_r)\overline{U}_b(\tau^B_r,0)\overline{U}_{a'}(\tau^A,\tau^A_s)U_c(t^B_r,t^A_s)\times\overline{U}_a(\tau^A_s,0)|\Psi\rangle. \tag{18}$$

Notice that the state $|\Psi(t)\rangle$ in Eq. (18) depends on t^A_0, t^A_s, t^B_0, and t^B_r through PTIs and through the term $U_c(t^B_r,t^A_s)$, defined in Eq. (8). As already discussed in the preceding section, the random phase φ_e present in Eq. (8) prevents Bob from recovering the CTI transit time $t^B_r - t^A_s$.

This example may be easily generalized to the case of multiple exchanges. Define t^A_h and t^B_h the times at which the last change in Alice and Bob's Hilbert space took place, i.e., the last time at which they either sent or received a signal. Expressing it in terms of the PTIs $\tau^A_h = t^A_h - t^A_0$ and $\tau^B_h = t^B_h - t^B_0$, the state of the system is then

$$|\Psi(t)\rangle = \overline{U}_A(\tau^A,\tau^A_h)\overline{U}_B(\tau^B,\tau^B_h)U_C(t,t_h)|\overline{\Psi}\rangle, \tag{19}$$

where A, B, and C refer, respectively, to the Hilbert spaces of Alice, Bob, and the exchanged system at time t, and t_h is the last time at which the Hilbert space of the exchanged system was modified. As can be seen by iterating Eq. (18), the state vector $|\overline{\Psi}\rangle$ in Eq. (19) depends only on PTIs and on the transit times of the systems Alice and Bob have exchanged. To show that the state $|\Psi(t)\rangle$ of Eq. (19) does not contain useful information to synchronize their clocks, suppose that (say) Bob performs a measurement at time t. The state he has access to is given by

$$\rho_B(t) = \mathrm{Tr}_{AC}[|\Psi(t)\rangle\langle\Psi(t)|] = \overline{U}_B(\tau^B,\tau^B_h)\mathrm{Tr}_{AC}[|\overline{\Psi}\rangle\langle\overline{\Psi}|]\overline{U}^\dagger_B(\tau^B,\tau^B_h), \tag{20}$$

where Tr_{AC} is the partial trace over $\mathcal{H}_C$ and $\mathcal{H}_A$ and where the cyclic invariance of the trace and the commutativity of operators acting on different Hilbert space has been used. The state $\rho_B(t)$ does not depend on τ^A. The only informations relevant to clock synchronization (that connect events on Alice's side to events on Bob's side) that may be recovered are the CTI transit times of the exchanged systems. However, in the case of complete dephasing ($\epsilon=1$), these quantities are irremediably spoiled by the random phases as discussed previously.

Up to now we have shown that by exchanging physical systems and performing a measurement, Alice and Bob cannot recover sufficient information to synchronize their clocks if the environment is completely dephasing. In other words, Alice can always encode some information on the system she sends Bob, but any operation she does, will always be referred to her PTI and will thus be useless to Bob if he ignores any CTI. That is equivalent to say that Alice may always send Bob some photographs of her clock, but Bob will have no use of them, since he cannot arrange them relative to his own time axis. A better strategy could be to measure only part of their systems and employ postselection schemes. As will be shown in the following section, even in this case all their efforts are in vain if hypothesis 2 applies.

B. Postselection schemes

Allow Alice and Bob to make partial measurements on their systems. The global system evolution is no longer unitary, since the measurements will project part of the Hilbert space into the eigenstates of the measured observable. The communication of the measurement results permits the implementation of postselection schemes. We will show that also in this case, Alice and Bob cannot synchronize their clocks in presence of dephasing in the communication channel.

Using the Naimark extension [10], one can assume the projective-type measurement as the most general. Suppose that Alice performs the first measurement at time t^A_m on a part of her system. Define $\mathcal{H}_{A_1}$ the Hilbert space that describes such a system, so that $\mathcal{H}_A = \mathcal{H}_{A_0}\otimes\mathcal{H}_{A_1}$ is the Hilbert space of Alice. The state of the system after the measurement for $t > t^A_m$ (and before any other measurement or system exchange) is

$$|\Psi(t)\rangle = U(t,t^A_m)P(A_1)|\Psi(t^A_m)\rangle, \tag{21}$$

where $|\Psi(t^A_m)\rangle$ is given in Eq. (19) and the global evolution operator is

$$U(t,t^A_m) = \overline{U}_A(\tau^A,\tau^A_m)\overline{U}_B(\tau^B,t^A_m - t^B_0)U_C(t,t^A_m) \tag{22}$$

with $\tau^A_m = t^A_m - t^A_0$. In Eq. (21) the measurement performed by Alice on $|\Psi(t^A_1)\rangle$ is described by the projection operator

$$P(A_1)|\psi\rangle \equiv \frac{1}{||\langle\psi|\phi\rangle_{A_1}||}(|\phi\rangle_{A_1}\langle\phi|\otimes\mathbb{1}_{A_0})|\psi\rangle, \tag{23}$$

where $\mathbb{1}_{A_0}$ is the identity on $\mathcal{H}_{A_0}$, $|\phi\rangle_{A_1}\in\mathcal{H}_{A_1}$ is the eigenstate relative to Alice's measurement result ϕ. Notice that Eqs. (21–23) take into account the postselection scheme in which Alice communicates her measurement result to Bob, since the operator $U(t,t^A_m)$ can depend on Alice's measurement result ϕ. Using again the commutation properties between operators that act on different spaces, Eq. (21) simplifies to

$$|\Psi(t)\rangle = \overline{U}_B(\tau^B,\tau^B_h)U_C(t,t_h)\overline{U}_A(\tau^A,\tau^A_m)\times P(A_1)\overline{U}_A(\tau^A_m,\tau^A_h)|\overline{\Psi}\rangle. \tag{24}$$

Equation (24) shows that even though the partial measurement introduces a nonunitary evolution term, this allows Alice to encode in the state only information about her PTI τ^A_m and nothing on the absolute time t^A_m (as measured by an external clock) or on any CTI. In fact, the same considerations of Eq. (20) apply and no information relevant to clock synchronization can be extracted from the state (24). The formalism introduced also allows one to consider the situation in which Alice does not look at her results (or does not communicate them to Bob): in this case, in Eq. (24) one must perform the sum on all the possible measurement results weighted by their outcome probability.

In the most general scenario Alice and Bob will perform multiple partial measurements, communicate by exchanging physical systems (as analyzed in the preceding section), and again perform partial measurements. By iterating Eq. (24) one can show that none of these efforts allows them to extract any CTI.

Before concluding, it is worth commenting on how the quantum clock synchronization scheme proposed in Ref. [3] is related to our analysis. In Ref. [3], the authors assume as a starting point that Alice and Bob share an entangled state of the form

$$|\chi\rangle=\sum_{a,b}\chi_{ab}|a\rangle|b\rangle, \tag{25}$$

where $|a\rangle$ and $|b\rangle$ are energy eigenstates of Alice's and Bob's systems, respectively, and where the sum on the indexes a and b runs over nondegenerate eigenstates. From the considerations given in the present section, one can show that, in the presence of a dephasing channel, such a state cannot be obtained starting from the initial state given in Eq. (3) without introducing some stochastic phases in it. For this reason, it cannot be obtained without relaxing hypothesis 2: such a protocol is then equivalent to classical protocols [4]. In fact, if one relaxes the hypotheses of channel dephasing, then it is possible to also achieve classical clock synchronization.

CONCLUSION

In conclusion, a definition of clock synchronization was given and it was shown that, under some very general hypotheses that preclude the possibility of employing classical protocols, such a synchronization is not possible. This does not imply that quantum mechanics may not be exploited in the clock synchronization procedures, but it may be limited only to enhancing classical clock synchronization protocols [6–8]. Indeed, we have shown elsewhere [8] that quantum mechanics may be used to cancel the effect of dispersion in clock synchronization.

ACKNOWLEDGMENTS

We wish to acknowledge support from DARPA Grant No. F30602-01-2-0546 under the QUIST program, ARO Grant No. DAAD19-001-0177 under the MURI program, and NRO Grant No. NRO-000-00-C-0158.

[1] A. Einstein, Ann. Phys. (Leipzig) **17**, 891 (1905).
[2] A. S. Eddington, *The Mathematical Theory of Relativity*, 2nd ed. (Cambridge University Press, Cambridge, UK, 1924).
[3] R. Jozsa, D.S. Abrams, J.P. Dowling, and C.P. Williams, Phys. Rev. Lett. **85**, 2010 (2000).
[4] E.A. Burt, C.R. Ekstrom, and T.B. Swanson, e-print quant-ph/0007030.
[5] J. Preskill, e-print quant-ph/0010098; U. Yurtsever and J. P. Dowling, Phys. Rev. A **65**, 052317 (2002).
[6] I.L. Chuang, Phys. Rev. Lett. **85**, 2006 (2000).
[7] V. Giovannetti, S. Lloyd, and L. Maccone, Nature (London) **412**, 417 (2001); V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. A **65**, 022309 (2002).
[8] V. Giovannetti, S. Lloyd, L. Maccone, and F.N.C. Wong, Phys. Rev. Lett. **87**, 117902 (2001).
[9] Consider, for example, a time dependent perturbation $H'_C(t)$ that acts on the system for a time δt during the interval $[t_s, t_r]$, taking the value $\hbar\Sigma_e P_e \nu_e$. Since $H'_C(t)$ is diagonal in the energy basis of H^o_C, then it is easy to derive the evolution dynamics (8).
[10] M.A. Naimark, Izv. Akad. Nauk USSR, Ser. Mat. **4**, 277 (1940); A. Peres, *Quantum Theory: Concepts and Methods*, Vol. 57 of *Fundamental Theories of Physics* (Kluwer Academic Publishers, Dordrecht, 1993).