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Modulation instability for a relaxational Kerr medium

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Abstract

We investigated on the influence of a temporally dispersive Kerr effect on the modulation instability and the propagation of solitary wave pulse. The modulation instability gain is derived and compared with numerical calculations. The role of nonlinear response time on reshaping the solitary pulse is examined.

1. Introduction

Modulation instabilities (MI) have important consequences for optical wave propagation in media with a nonlinear response. The collapse and break-up of a pulse into sub pulses and broadening of the spectrum has been exploited for a number of applications, including white light generation and the frequency comb for metrology [1]. Under the right circumstances the modulation instability (MI) is deeply connected with optical solitons, such as, found for the nonlinear Schrodinger's equation (NLSE) [2]. The NLSE's, in different forms, can describe different types of soliton propagation in nonlinear medium. The simplest (1+1)-dimensional NLSE demonstrates that the temporal soliton is a stable entity that the group velocity dispersion (GVD) is balanced by self-phase modulation (SPM) in the nonlinear Kerr medium, meaning that the index of refraction changes with the wave's intensity [3].

The MI has been observed in many nonlinear systems such as plasma physics and fluid dynamics [4–6]. In the linearized regime, MI refers to the exponential growth of a small perturbation in the medium, which can break a continuous wave into periodic train of soliton-like pulses [7]. Under certain circumstances MI can be also interpreted as a degenerated four-wave mixing process in the frequency domain. MI was first studied with coherent light beam in one dimensional self-focusing medium [8]. Later research shows that MI is also found in incoherent beams [9]. Recent studies show that spatial MI can exist in both self-focusing and self-defocusing medium if the nonlinearity is non-instantaneous [10,11], which implies new applications in the nonlinear medium like CS_2 [12,13]. Temporal MI in directional couplers with relaxing media was examined by Trillo et al. [14]. This paper has results that are directly related to those given here. The spatial–temporal MI of counter-propagating waves was also investigated, implying potential applications in optical limiting [15,16].

In this paper, the finite response time of the Kerr effect is taken into consideration. The NLSE is modified by introducing a single temporal relaxing term to modify the refractive contribution to the NLSE. The modified MI is derived from our equations. Using the dynamical model, the modulation instability growth is verified and solitary wave propagation is numerically studied in detail.

2. Model

The propagation of optical soliton in instantaneous nonlinear medium can be described by the nonlinear Schrodinger's equation (NLSE). When perturbations are ignored, the equation takes the form

$$\mathrm{i}\frac{\partial E}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 E}{\partial t^2} - \gamma \mid E|^2 E \tag{1}$$

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where $E = E\,(t, z)$ describes the electric field envelope in both time and space. β_2 is the group velocity dispersion (GVD) parameter, and γ is nonlinear coefficient for self-phase modulation (SPM). This equation provides simple insights into the dynamics of optical solitons. The first term on the right of equation stands for the GVD of the pulse, which alone broadens the optical pulse. The second term represents the nonlinear Kerr effect, which can compress the pulse. When the GVD and the SPM effects exactly balance each other, an optical soliton can be generated that travels without shape distortion.

For non-instantaneous, nonlinear media, because of the delay in the response time in such material/structures, perturbations of the NLSE are added and the dynamical behavior is affected. A soliton-like pulse propagating in a one dimensional Kerr medium, which has a non-instantaneous nonlinear property characterized by the response time τ, will undergo shape distortions on a length scale that is determined by the strength of the perturbation. In this paper we expand the NLSE to a set of coupled equations.

$$\mathrm{i}\frac{\partial E}{\partial z} = \frac{1}{2}\frac{\partial^2 E}{\partial t^2} + NE \tag{2a}$$

$$\frac{\partial N}{\partial t} = \frac{1}{\tau}(-N + |E|^2) \tag{2b}$$

For the sake of simplicity, the β_2 and γ in Eq. (1) are scaled in the new model. $N = N\,(t, z)$ represents the nonlinear index of the medium. The medium dynamics is described by a simple relaxational model in Eq. (2b), sometimes called the Debye relaxation model [12,13]. The parameter τ is the medium's response time. The parameter N replaces the original NLSE nonlinearity term $|E|^2$. The dynamics of N is related to the local field intensity and the response time of the Kerr medium.

3. Modulation instability

The modulation instability, which exists in many nonlinear systems, refers to the phenomenon that a weak perturbation from the steady-state solution can grow exponentially with propagation distance. In the NLSE case the modulation instability is a result of the interplay between the GVD and the Kerr effect of the medium.

The steady-state solution to Eq. (2) for E possesses the form of a continuous wave

$$E = E_0 e^{\mathrm{i}|E_0|^2 z} \tag{3}$$

One key question is whether this continuous wave is stable against small perturbations. To testify, we add small perturbations e to E

$$E_p = (E_0 + e)e^{\mathrm{i}|E_0|^2 z} \tag{4}$$

The perturbation of N from its steady-state value is expressed in a linear form as

$$N_p = N_0 + n \tag{5}$$

where n is the perturbation from N_0. Note that e can be complex.

We begin the stability analysis by substituting Eqs. (4) and (5) into Eq. (2) with the linearized result being

$$\mathrm{i}\frac{\partial e}{\partial z} = \frac{1}{2}\frac{\partial^2 e}{\partial t^2} + nE_0, \tag{6a}$$

$$\frac{\partial n}{\partial t} = \frac{1}{\tau}(-n + (e + e^*)E_0). \tag{6b}$$

It's simple to solve Eq. (6) in the frequency domain. Taking the Fourier transform and eliminating the function $\tilde{n}$, the fields are expressed in coupled equations as

$$\left(k + \frac{\Omega^2}{2} - \frac{E_0^2}{1 + \mathrm{i}\Omega\tau}\right)e(\Omega, k) - \frac{E_0^2}{1 + \mathrm{i}\Omega\tau}e^*(-\Omega, -k) = 0 \tag{7a}$$

$$-\frac{E_0^2}{1 + \mathrm{i}\Omega\tau}e(\Omega, k) + \left(-k + \frac{\Omega^2}{2} - \frac{E_0^2}{1 + \mathrm{i}\Omega\tau}\right)e^*(-\Omega, -k) = 0 \tag{7b}$$

The nontrivial solutions for both $e(\Omega, k)$ and $e^*(-\Omega, -k)$, leads to the following dispersion relation between k and Ω (Eq. (13a) in Ref. [15]).

$$k = \pm\sqrt{\frac{\Omega^4}{4} - \frac{\Omega^2 E_0^2}{1 + \Omega^2\tau^2}(1 - \mathrm{i}\Omega\tau)} \tag{8}$$

The frequency dependent dispersion and gain coefficients are extracted from this result. The gain is determined by the imaginary part of the wave vector k. For comparison the well-known gain spectrum for the instantaneous nonlinear medium, the dispersion relation between k and Ω is [2]

$$g(\Omega) = \frac{|\,\Omega\,|\sqrt{\Omega_c^2 - \Omega^2}}{2} \tag{9}$$

where $\Omega_c = 2|E_0|$.

4. Results

The gain spectrum of instantaneous Kerr medium for $|E_0| = 1$ is plotted in Fig. 1. The actual frequency and wave

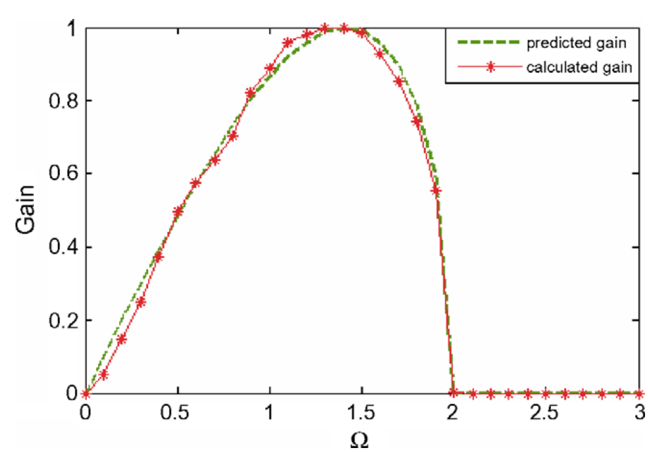

Fig. 1. Gain spectrum for the instantaneous nonlinear Kerr medium.

vector of the perturbation are shifted from the carrier frequency and wave number values $\omega_0 + \Omega$ and $\beta_0 + k$, respectively. The points in Fig. 1 are the numerical values of the gain calculated by solving the NLSE equation with a small sinusoidal perturbation.

As seen in Eq. (9) and illustrated in Fig. 1 for the instantaneous Kerr medium response, the peak gain occurs at $\Omega = \pm\frac{\Omega_c}{\sqrt{2}}$ with a value $g_{\max} = g\left(\pm\frac{\Omega_c}{\sqrt{2}}\right) = 1$. The gain only occurs in the regime $|\Omega| < \Omega_c$, indicating that temporally homogeneous wave modulated by noise experiences exponential growth only within the cutoff frequency region. For $|\Omega\ | > \Omega_c$, k is always a real number, meaning that the steady-state solution is stable against any perturbation. The gain curve from our simulation agrees with the analytical result.

For non-instantaneous Kerr medium, $\tau \neq 0$, the modified NLSE are solved. From Eq. (8), k remains complex in the entire frequency domain. The real part of k indicates that the wave always propagates with certain periodicity $Z = \frac{2\pi}{\mathrm{Re}(k)}$. The imaginary part of k shows that during propagation, the amplitude of the perturbation increases exponentially with distance. Put into equation,

$$\left|\frac{e(z)}{e(0)}\right| = \exp(g(\Omega)z), \quad g(\Omega) = \mathrm{Im}(k) \tag{10}$$

The propagation of a small perturbation in the non-instantaneous nonlinear medium can be interpreted as a combination of periodically repetitive propagation and exponential growth of perturbation amplitude at the same time. We choose sinusoidal perturbation at $z = 0$, i.e. $e(t) = \varepsilon\cos(\Omega t)$, in our numerical experiment, where ε is a small parameter and the frequency Ω is varied. Fig. 2 illustrates the growth of a harmonic perturbation in the non-instantaneous nonlinear medium. Three typical relaxational response parameters chosen for further analysis are: $\tau_1 = 0.2$, $\tau_2 = 1$ and $\tau_3 = 10$. By analyzing the variance in the perturbation amplitude with propagation distance, the gain can be extracted. Using different perturbation frequencies the gain spectra are calculated.

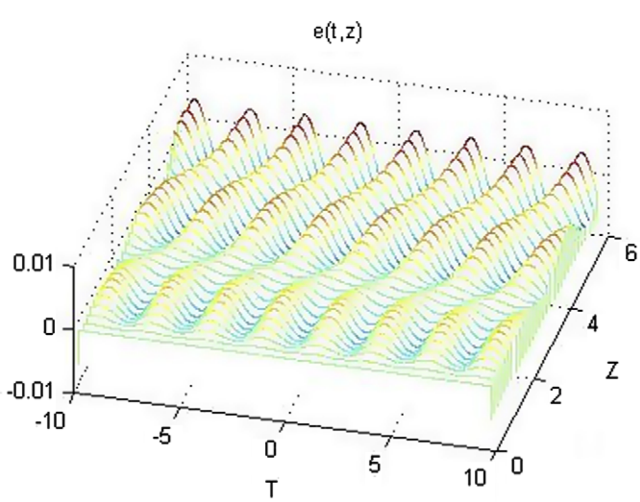

Fig. 2. Gain of the periodic temporal perturbation versus propagation distance. Perturbation parameters: $\varepsilon = 0.001$ and $\Omega = 2.5$.

The graphs of the gain versus frequency in Figs. 3–5 demonstrate that there is a broader gain bandwidth for the non-instantaneous nonlinear medium and the peak gain is reduced as the relaxation time increases. There is excellent agreement between the theoretical predicted gain curve and our numerical simulation of the periodic amplitude gain. Only the positive frequency side of the gain curve is shown here; the spectra are symmetric around $\Omega = 0$.

As noted the peak gain decreases, as the relaxing time τ increases. This can be interpreted by treating the modulation instability as a four-wave mixing process, the perturbation being the probe waves and the CW pump beam. The energy of the two photons from the pump produces two new conjugate frequency photons. For a specific frequency pair $\pm\Omega$, the gain coefficient is roughly reduced by the term $1 + (\Omega\tau)^2$ (see Eq. (8)). In the time domain for small relaxation times, a term similar to the Raman contribution is added and the pulse undergoes a redistribution of energy from the pump frequency to lower frequencies. At higher frequencies the gain is reduced because of weaker coupling between the waves.

Comparing the gain curves of the instantaneous nonlinear medium and its counterpart, the gain peak is shifted from $\pm\Omega_c/\sqrt{2}$ toward the carrier frequency. The reason for this phenomenon is that the relaxing time τ depresses the gain at low and intermediate frequencies ($|\Omega| < 2$). On the other hand the cutoff at $|\Omega| = 2$ when $\tau = 0$ is relaxed because the marginal instability has been replaced by a pair of solutions giving gain and loss and both are excited by a general initial condition.

5. Soliton-like pulse propagation

With the modified model, the influence of the non-instantaneous Kerr medium on the solitary wave propagation can be studied. One popular numerical method used to

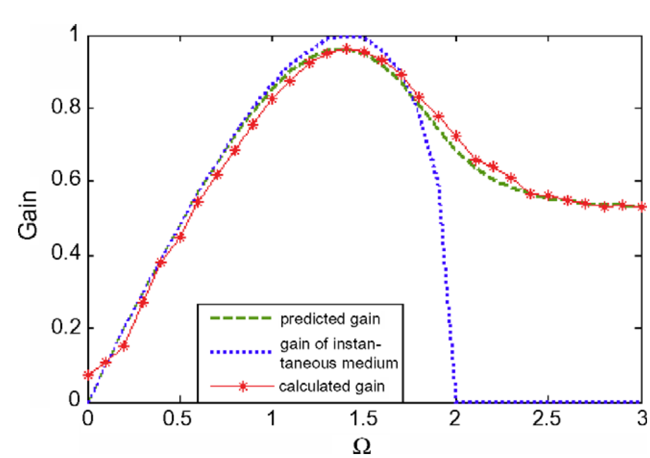

Fig. 3. The gain spectrum for $\tau_1 = 0.2$.

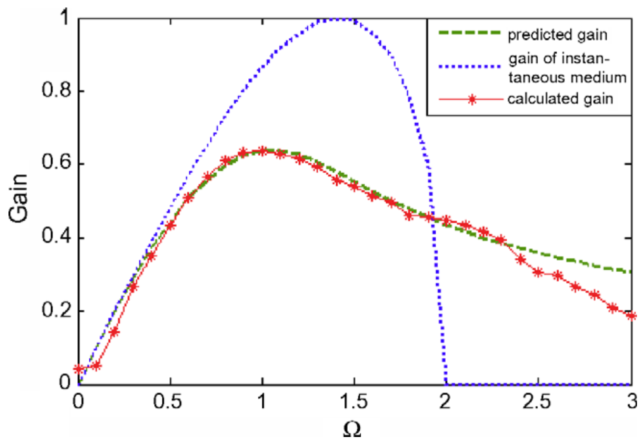

Fig. 4. The gain spectrum for $\tau_2 = 1$.

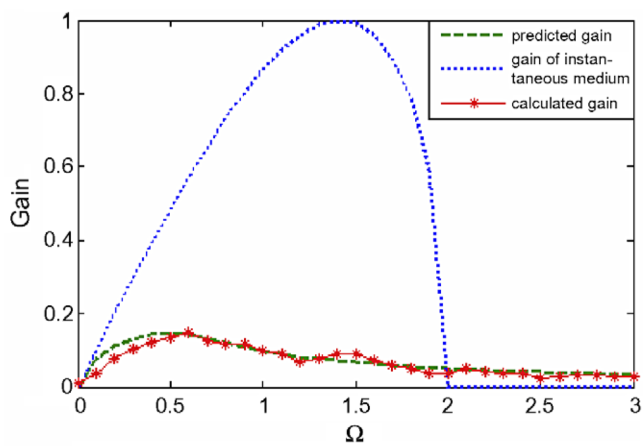

Fig. 5. The gain spectrum for $\tau_3 = 10$.

solve Eq. (2a) is called the split-step or spectral method [17,18]. In this method the linear and nonlinear parts of the partial–differential equation are separated and handled in separate steps. By doing so, we rewrite Eq. (2a) in the following form

$$\frac{\partial E}{\partial z} = -\frac{\mathrm{i}}{2}\frac{\partial^2 E}{\partial t^2} - \mathrm{i}NE = [\tilde{D} + \tilde{N}]E \tag{11}$$

where $\tilde{D} = -\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial t^2}$ and $\tilde{N} = -\mathrm{i}N$ are the linear and the nonlinear operator, respectively.

The linear step of this method can be easily done in the frequency domain and the nonlinear step can be treated in the time domain. Fast Fourier transform methods are applied to solve the dynamical equations. The second-order solution for E at incremented distance $z + \Delta z$ is found to be

$$E(t, z + \Delta z) = \exp\left(\frac{\Delta z D}{2}\right)\exp(\Delta z N)\exp\left(\frac{\Delta z D}{2}\right)E(t, z) \tag{12}$$

The precision of the numerical solution depends on both the time-frequency domain resolutions and the step sizes along the propagation direction.

The integral form solution of Eq. (2b) is

$$N(t, z) = \frac{1}{\tau}\int_{-\infty}^{t} \mathrm{e}^{-\frac{(t-t')}{\tau}} \mid E(t', z)|^2 \mathrm{d}t' \tag{13}$$

Eqs. (12) and (13) constitute the mathematical framework for our computer calculations.

We examine the effect of non-instantaneous Kerr mediums on the propagation of the soliton-like pulse with a hyperbolic secant intensity profile in time, that is,

$$E(t) = \mathrm{sech}(t) \tag{14}$$

This is the fundamental soliton solution of the instantaneous NLSE.

Again, for our illustration of the perturbation effects, three typical relaxational response parameters are chosen $\tau_1 = 0.2$, $\tau_2 = 1$ and $\tau_3 = 10$. Comparison between Figs. 6–8 leads to several observations about the affect of the non-instantaneous nonlinear medium. The graphs of the function N, show the delayed response that develops in the medium yielding a persistent tail as the pulse passes a given position. The pulse peaks are delayed to longer times and eventually the pulse envelope is severely broadened. Analysis of the pulse spectrum shows that its peak is shifted to higher frequencies. The process is analogous to the red-shifted frequency shift found in intra-pulse stimulated Raman scattering. The pulse delay is more evident for the shorter relaxation times and the effect of a finite relax-

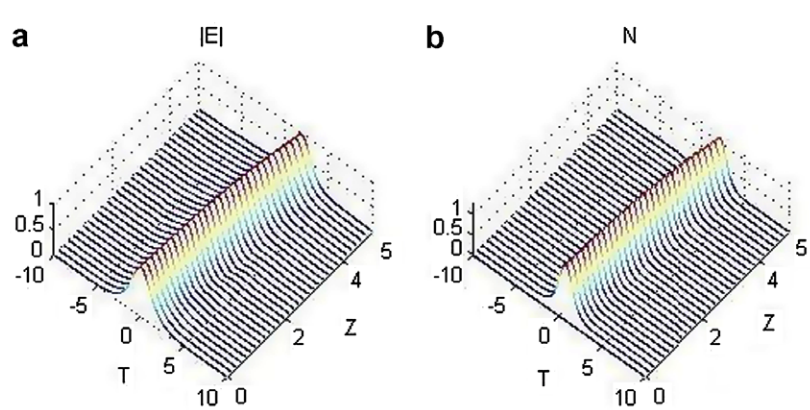

Fig. 6. $\tau = 0.2$ (a) pulse shape in the medium (b) N of the medium. We follow the propagation to $z = 5$.

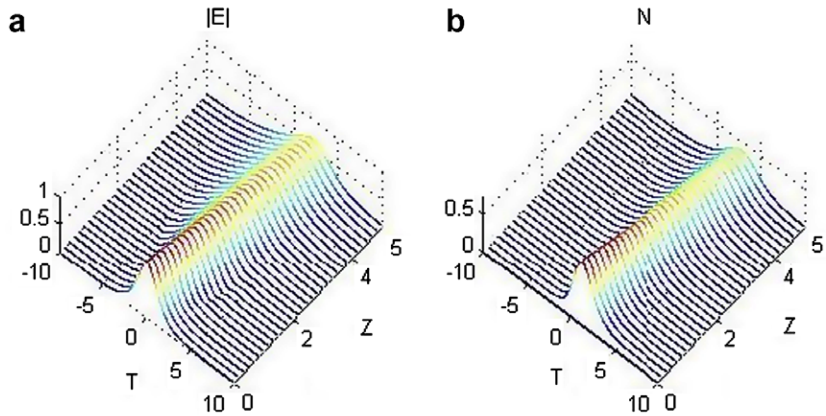

Fig. 7. $\tau = 1$ (a) pulse shape in the medium (b) N of the medium.

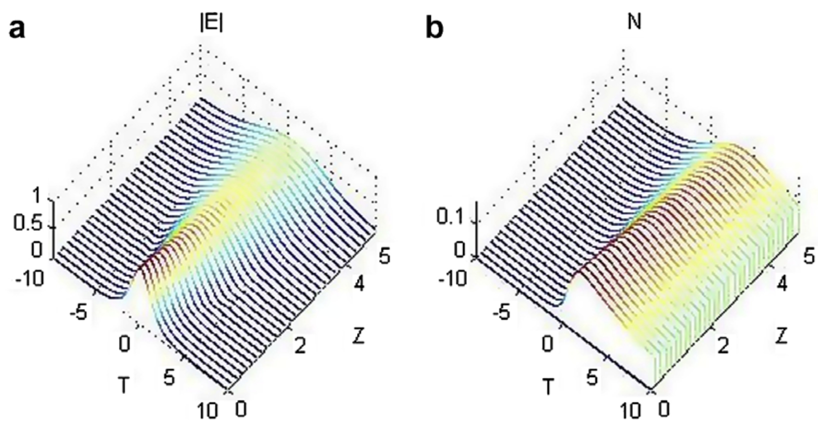

Fig. 8. $\tau = 10$ (a) pulse shape in the medium (b) N of the medium.

ation time is already apparent for $\tau = 0.2$. As τ increases, the magnitude of N decreases, and the effect of the nonlinearity is diminished, which also limits the frequency shift. The balance between the nonlinearity and the dispersion cannot be maintained over distances that are sufficiently long.

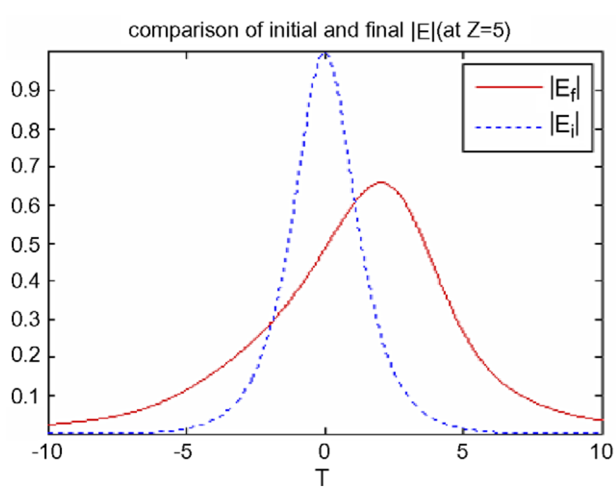

Fig. 9. Pulse spreading in non-instantaneous Kerr medium ($\tau = 1$) with the initial field $|E_i|$ and the final field, $|E_f|$.

This is intuitively understandable because τ also determines the magnitude of the nonlinear response in our model. As stated earlier, the optical soliton is a dynamical entity derived from the balance between GVD and SPM. When the Kerr effect becomes weaker, it no longer balances the dispersion and the pulse spreads. Fig. 8 shows that for a soliton-like pulse propagating in non-instantaneous Kerr medium ($\tau = 1$), after certain distance ($z = 5$), the maximum of N drops to about 40% of that for the instantaneous nonlinear medium. The pulse, unable to maintain its shape broadens in an asymmetric way as shown in Fig. 9, where the initial and final field envelope pulse profiles are shown. This pulse spreading process, in return, leads to even weaker Kerr effect along the propagation.

6. Conclusion

In this paper, we presented the effect of non-instantaneous Kerr medium on the MI and showed it's affect on the propagation of a soliton-like pulse. We showed that the finite response time of Kerr effect alters the MI by lowering the gain and removing the gain cutoff frequency. It was analytically shown and numerically demonstrated that a long relaxing time of the medium diminishes the nonlinear Kerr effect and shifts the frequency, resulting in a failure to maintain the shape of the soliton-like pulse. Compared with the ideal Kerr medium, the new gain bandwidth is no longer limited to a frequency region below the cutoff frequency, Ω_c. For $\Omega < \Omega_c$, the gain decreases because fewer photons are generated by the four-wave mixing procedure due to the smaller effective nonlinearity. There is a crossover around Ω_c where the tail of the gain curve extends to high frequencies. For $\Omega > \Omega_c$, unlike the instantaneous Kerr medium, the MI gain region is extended to break-up temporal pulses at higher frequencies. The soliton-like pulses undergo a frequency shift, similar to that found for stimulated Raman scattering, which delays the pulses to later times.

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