





where  $\phi$  is the round-trip phase shift,  $\nu$  is the lasing frequency, and  $\Omega$  is the resonant frequency of the cavity in the absence of the medium.  $\Omega$  is given by  $2\pi mc/L$  where  $L$  is the length of the cavity,  $c$  is the vacuum speed of light, and  $m$  is the mode number.  $\chi'$  and  $\chi''$  are the real and imaginary part of the susceptibility of the medium, respectively.  $E$  is the field amplitude, and  $Q$  is the cavity quality factor.  $\nu_0$  is the frequency around which  $\chi''$  is symmetric.  $\Omega = \nu_0$  for a particular length of the cavity:  $L = L_0$ . In our model,  $L$  will be allowed to deviate from  $L_0$ , thereby making  $\Omega$  differ from  $\nu_0$ . For simplicity, we assume a situation where lasing is unidirectional, made possible by the presence of an optical diode inside the cavity. Any loss induced by the diode is included in  $Q$ .

For convenience, we define the parameters,  $\Delta \equiv \Omega - \nu_0$ ,  $\delta \equiv \nu - \nu_0$ . The derivatives  $d\Delta/dL$  and  $d\delta/dL$  characterize the resonant frequency shifts under a perturbation of  $L$  in the empty and filled cavity, respectively. We consider the ratio,  $R \equiv (d\delta/dL)/(d\Delta/dL)$  to determine if the amount of the frequency shift is enhanced ( $R > 1$ ) or diminished ( $R < 1$ ) by the intracavity medium. To derive  $R$ , we begin with Eq. (1a). In steady state ( $\dot{\phi} = 0$ ) subtracting  $\nu_0$  from both sides, differentiating with respect to  $L$ , and applying  $d\nu = d\delta$ , we get  $d\delta/dL + \chi'/2(d\delta/dL) + \nu/2(d\chi'/dL) = d\Delta/dL$ . By substituting  $(d\chi'/d\delta)(d\delta/dL)$  for  $d\chi'/dL$ , we get:

$$R = 1 / \left[ 1 + \frac{\chi'}{2} + \frac{\nu}{2} \frac{d\chi'}{d\nu} \right] \quad (2)$$

In order to determine the value of  $R$ , it is necessary to know how  $\chi'$  depends on  $\nu$ . In case of a passive cavity, this would be determined solely by the response of the medium to a weak probe, and would be related to  $\chi''$  by the KK relations. In the case of an active cavity, however, this is no longer true, since  $\chi'$  depends indirectly on  $E$ , which varies as a function of  $\nu$ , as determined by a self-consistent solution of Eq. (1). In the later parts of the paper, we explore this interdependence of  $\chi'$  and  $E$  in detail in order to establish the behavior of the superluminal laser.

We can at this point develop additional insight into the behavior of  $R$  by simply assuming that  $\chi'$  is antisymmetric around  $\nu = \nu_0$  (to be validated later). We can then expand  $\chi'$  around  $\nu = \nu_0$ , keeping terms up to  $(\Delta\nu)^3$  where  $\Delta\nu = \nu - \nu_0$ . We get  $\chi' = \chi'(\nu_0) + \Delta\nu\chi'_1 + (\Delta\nu)^3\chi'_3/6$  where  $\chi'_n \equiv (d^n\chi'/d\nu^n)|_{\nu=\nu_0}$  ( $n = 1$  or  $3$ ). Since  $\chi'$  is assumed to be antisymmetric around  $\nu_0$ , we have  $(d^2\chi'/d\nu^2)|_{\nu=\nu_0} = 0$ . Differentiating  $\chi'$  with respect to  $\nu$ , and inserting  $\chi'$  and the derivative of  $\chi'$  in Eq. (2), we get:  $R \equiv 1 / \left[ 1 + \chi'(\nu_0)/2 + \nu_0\chi'_1/2 + \nu_0\chi'_3(\Delta\nu)^2/4 \right]$ . Here, we have dropped terms that are small due to the fact that  $|\Delta\nu| \ll \nu_0$ . Note that if  $\chi'$  is assumed to be linear, corresponding to keeping only the first two terms in the expansion of  $\chi'$ , the enhancement factor simplifies to:  $R \equiv 1 / \left[ 1 + \chi'(\nu_0)/2 + \nu_0\chi'_1/2 \right]$ .

To illustrate the physical meaning of this result, we note first that the index can be expressed as  $n(\nu) = [1 + \chi']^{1/2} \cong 1 + \chi'/2$ . The group index can then be written as  $n_g = n_0 + \nu_0 n_1 = 1 + \chi'(\nu_0)/2 + \chi'_1\nu_0/2$ , where  $n_0 = n(\nu_0)$  and  $n_1 \equiv (dn/d\nu)|_{\nu=\nu_0}$ . Thus, the enhancement factor in this linear limit is given simply as  $R \cong 1/n_g$ . For normal dispersion ( $n_g > 1$ ),  $R$  becomes less than one. Hence, the resonant frequency shift with respect to the length variation decreases compared to the shift in an empty cavity. For anomalous dispersion

( $n_g < 1$ ), the frequency shift is amplified to be equal to  $1/n_g$  times the amount of the shift in the empty cavity.

In order to determine the actual value of  $R$  for an active cavity, we need first to establish the explicit dependence of  $\chi'$  on the lasing frequency,  $\nu$ . To this end, we first solve Eq. (1b) in steady state ( $\dot{E} = 0$ ) so that  $\chi''(E, \nu) = -1/Q$  for  $E \neq 0$ . Since  $\chi''$  is a function of  $E$  and  $\nu$ , the solution to the equation yields the saturated electric field  $E$  in steady state inside the laser cavity as a function of the lasing frequency  $\nu$ .

Let us consider the case in which the cavity contains a medium with a narrow absorption as well as a medium with a broad gain. This configuration creates a gain profile with a dip in the center. The imaginary part of the susceptibility  $\chi''$  can then be written as:

$$\chi'' = -\frac{G_e \Gamma_e^2}{\mathcal{G}_e} + \frac{G_i \Gamma_i^2}{\mathcal{G}_i} \quad (3a)$$

where  $\mathcal{G}_k = 2\Omega_k^2 + \Gamma_k^2 + 4(\nu - \nu_0)^2$ , ( $k = e$  or  $i$ ). We use the subscript “e” for the “envelope” gain profile and “i” for the narrower absorption profile.

Using the modified Kramers-Kronig (MKK) relation [12,13] for the saturated susceptibility, we can then express the real part of the susceptibility  $\chi'$  as:

$$\chi' = \frac{2G_e(\nu - \nu_0)\Gamma_e}{\mathcal{G}_e} - \frac{2G_i(\nu - \nu_0)\Gamma_i}{\mathcal{G}_i} \quad (3b)$$

Here  $\Omega_i$  ( $\Omega_e$ ) is the Rabi frequency equal to  $\wp_i E/\hbar$  ( $\wp_e E/\hbar$ ) where  $\wp_i$  ( $\wp_e$ ) is the dipole moment associated with the absorbing (amplifying) medium. The gain and the absorption linewidths are denoted by  $\Gamma_e$  and  $\Gamma_i$ , respectively. Using the Wigner-Weisskopf model [16] for spontaneous emission, we can define two parameters:  $\xi_i \equiv \wp_i^2/(\hbar^2 \Gamma_i)$  and  $\xi_e \equiv \wp_e^2/(\hbar^2 \Gamma_e)$ . In terms of these parameters, the Rabi frequencies can then be expressed as  $\Omega_i^2 \equiv \Gamma_i E^2 \xi_i$  and  $\Omega_e^2 \equiv \Gamma_e E^2 \xi_e$ . The gain parameters can then be expressed as  $G_i \equiv \hbar N_i \xi_i / \varepsilon_0$  and  $G_e \equiv \hbar N_e \xi_e / \varepsilon_0$ , where  $\varepsilon_0$  is the permittivity of free space, and  $N_e$  and  $N_i$  represent the density of quantum systems for the absorptive and the amplifying media, respectively.

It is instructive to consider first the case of a conventional gain medium, by setting  $G_i = 0$ . From Eq. (3),  $\chi'$  and  $\chi''$  are then simply related to each other as  $\chi'/\chi'' = -2(\nu - \nu_0)/\Gamma_e$ . From  $\chi'' = -1/Q$ , it then follows that  $\chi' = 2(\nu - \nu_0)/(Q\Gamma_e)$ . The expression for  $\chi'$  is linear in  $\nu$ , and antisymmetric around  $\nu_0$ . Note that according to Eq. (1b),  $Q = \nu_0/\Gamma_c$ , where  $\Gamma_c$  is the empty cavity linewidth. Therefore, we find from  $R$  in the linear limit that  $R = 1/n_g$ , where  $n_g = 1 + \Gamma_c/\Gamma_e$ . Since  $n_g > 1$ , this implies a reduction in sensitivity when compared to an empty cavity. In a typical laser,  $\Gamma_c/\Gamma_e$  is very small so that this reduction is rather insignificant. The behavior of such a system in the context of the KK relations is discussed in detail in Ref.14, which also addresses the inadequacies of previous studies.

For the case of a conventional laser medium discussed above, it was easy to determine the value of  $\chi'$  because of the simple ratio between  $\chi'$  and  $\chi''$ . In particular, this ratio does not depend on the laser intensity. However, for the general case (i.e.  $G_i \neq 0$ ), the two terms in  $\chi''$  are highly dissimilar. As such, it is no longer possible to find a ratio between  $\chi'$  and  $\chi''$  that is independent of the laser intensity. Thus, in this case, we need to determine first the manner in which the laser intensity depends on all the parameters, including  $\nu$ . We define  $I \equiv |E|^2$  so that  $\Omega_i^2 \equiv \Gamma_i I \xi_i$  and  $\Omega_e^2 \equiv \Gamma_e I \xi_e$ . By setting Eq. (3a) equal to  $-1/Q$ , we get  $aI^2 + bI + c = 0$ , where  $a$ ,  $b$ , and  $c$  are function of various parameters. We keep the solution that is positive over the

lasing bandwidth:  $I = (-b + \sqrt{b^2 - 4ac}) / (2a)$ . Substituting this solution for  $I$  to evaluate  $\Omega_e^2$  and  $\Omega_c^2$ , in Eq. (3b) we get an analytic expression for  $\chi'$ .

The behavior of  $\chi'$  can be understood by plotting it as a function of  $\nu$ . For illustration, we consider  $\Gamma_e = 2\pi \times 10^9 \text{ s}^{-1}$ ,  $\Gamma_i = 2\pi \times 10^7 \text{ s}^{-1}$ ,  $\nu_0 = 2\pi \times 3.8 \times 10^{14} \text{ s}^{-1}$ ,  $N_e = 9 \times 10^6$ , and  $N_i = 1.2645 \times 10^{11}$ . To fulfill the lasing condition in the spectral range of the absorption dip, we consider the gain peak  $G_e = 12/Q$  and the absorption depth  $G_i = (10 + \varepsilon)/Q$  so as to ensure  $G_e - G_i > 1/Q$  at  $\nu = \nu_0$  where  $\varepsilon$  is a small fraction. For a given value of  $G_e$ , the choice of  $\varepsilon$  is critical in determining the optimal behavior of  $\chi'$ . The particular choice of  $\varepsilon = 0.11591$  was arrived at via a simple numerical search through the parameter space. Figure 1 shows  $\chi'$  as a function of  $\nu$ . Note first that far away from  $\nu = \nu_0$ , it agrees asymptotically with the linear value of  $\chi'$  for the case of  $G_i = 0$ , indicated by the dotted line. The inset figure shows an expanded view of  $\chi'$ . The steep negative slope of  $\chi'$  around around  $\nu = \nu_0$  is the feature necessary to produce the fast light effect ( $n_g \approx 0$ ).

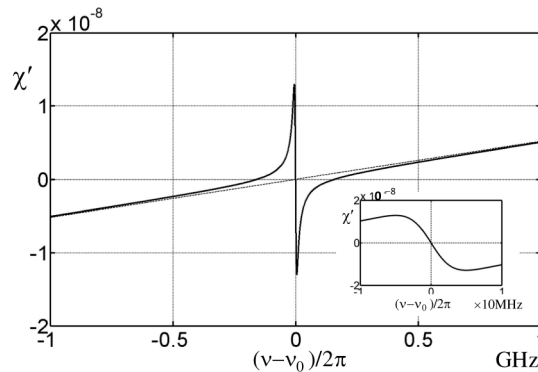


Fig. 1. Real part of the steady-state susceptibility for the combined absorbing and amplifying media (solid line), and the conventional gain medium (dashed line). The inset shows an expanded view of the solid line in the main figure

We can now evaluate the enhancement factor,  $R$ , as expressed in Eq. (2). We note first that  $d\chi'/d\nu = \partial\chi'/\partial\nu + (\partial\chi'/\partial I)(\partial I/\partial\nu)$ , where  $\partial I/\partial\nu$  is evaluated from the solution of the quadratic equation for  $I$ , and  $\partial\chi'/\partial\nu$  and  $\partial\chi'/\partial I$  are evaluated from Eq. (3b). Inserting  $\partial\chi'/\partial\nu$  and Eq. (3b) in Eq. (2), we find  $R$  as a function of  $\nu$ . This is shown in Fig. 2 for the parameters mentioned above. The insets (a) and (b) show a view expanded horizontally and a view on a linear vertical scale, respectively. The enhancement reaches a peak value of  $\sim 1.8 \times 10^5$  at the center of the gain dip [inset (a)], drops to a minimum ( $\sim 0.89$ ) and increases to a value close to unity [inset (b)]. These attributes are consistent with the behavior of  $\chi'$  shown in Fig. 1. The peak value of  $R$  corresponds to the steep negative dispersion. As the dispersion turns around and becomes positive, the value of  $R$  drops significantly below unity. Finally, as the dispersion reaches the weak, positive asymptotic value, we recover the behavior expected of a conventional laser, with  $R$  being very close to, but less than unity [inset (b)].

The black lines shown in Fig. 2 correspond to the exact value of  $R$  given by Eq. (2). It is also instructive to consider the approximate values of  $R$ , where we assume  $\chi'$  to be linear. This is shown by the dotted line in inset (a). Of course, this linear approximation is valid only over a very small range around  $\nu = \nu_0$ . However, it does show clearly that the peak value of  $R$  can be understood simply to be due to the linear, negative dispersion at  $\nu = \nu_0$ .

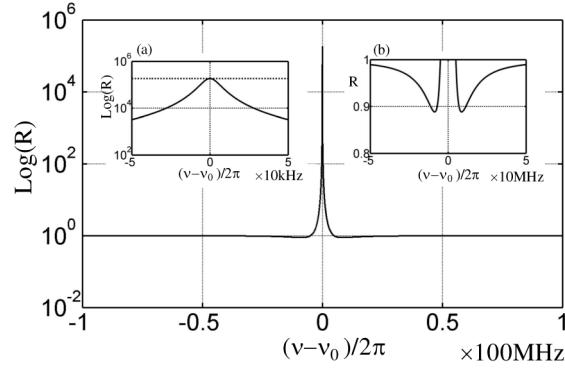


Fig. 2. Sensitivity enhancement associated with Eq. (2). The inset (a) shows  $R$  in Eq. (2) in an expanded view (solid line), and its value in the linear limit (dotted line). The inset (b) is an expanded view of  $R$  with a linear vertical scale.

The enhancement factor  $R$  indicates how the lasing frequency  $\nu$  changes when the length of the cavity,  $L$ , is changed. It is also instructive to view graphically the dependence of  $\nu$  on  $L$  explicitly. As discussed earlier, in the empty cavity, we have  $\nu_0 = 2\pi mc/L_0$ , where  $\nu_0$  is the resonance frequency (chosen to coincide with the center of the dispersion profile). For concreteness, we have used  $\nu_0 = 2\pi \times 3.8 \times 10^{14}$ , corresponding to the  $D_2$  transition in Rubidium atoms. We now choose a particular value of the mode number  $m$ , so that  $L$  is close to one meter. Specifically  $m = 1282051$  yields  $L_0 = 0.99999978$  meter.

When the dispersive gain medium is present, the lasing frequency will be  $\nu = \nu_0$  if  $L$  is kept at  $L_0$ . If  $L$  deviates from  $L_0$ , then  $\nu$  changes to a different value. Specifically,  $L = 2\pi mc/\Omega$ , and  $\nu = \Omega/(1 + \chi'/2)$  from Eq. (1a) in steady state, so that  $L = (2\pi mc/\nu)/(1 + \chi'/2)$ . Using the dependence of  $\chi'$  on  $\nu$  as shown in Fig. 1, we can thus plot  $L$  as a function of  $\nu$ , as shown in Fig. 3. Even though  $L$  is plotted on the vertical axis, this plot should be interpreted as showing how  $\nu$  changes as  $L$  is varied. For  $\nu$  far away from  $\nu_0$ , the variation of  $\nu$  as a function of  $L$  is essentially similar to that of an empty cavity, indicated by the dotted line.

Around  $\nu = \nu_0$ , a small change in  $L$  corresponds to a very big change in  $\nu$  as displayed in the inset figure. Specifically, we see that  $\Delta L \approx 10^{-13}$  meter produces  $f \equiv \Delta\nu/(2\pi) \approx 10^5$  Hz, corresponding to  $\Delta f/\Delta L \sim 10^{18}$ . In contrast, for a bare cavity,  $\Delta L \approx 2 \times 10^{-7}$  produces  $\Delta\nu/(2\pi) \approx 8 \times 10^7$  Hz, corresponding to  $\Delta f/\Delta L \sim 4 \times 10^{14}$ . The value of  $\Delta f/\Delta L$  for a conventional laser is also very close to this value, as indicated by the convergence of the dotted and solid lines for  $\nu$  far away from the superluminal region. Thus the enhancement factor,  $R$ , is  $\sim 2.5 \times 10^3$ . If we zoom in even more, we will eventually see that as  $\Delta L \rightarrow 0$ , we have  $R \sim 1.8 \times 10^5$ , as shown in Fig. 2.

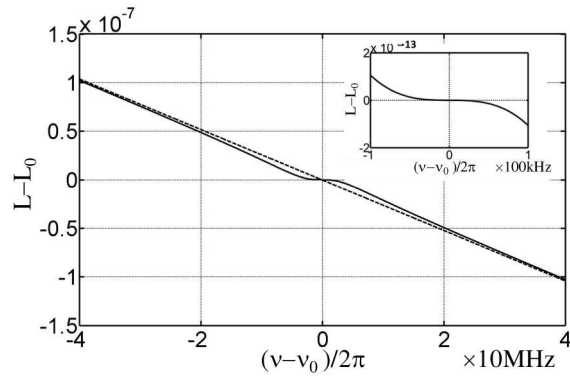


Fig. 3. Illustration of relation between  $L-L_0$  and  $v-v_0$  for  $|v-v_0|/2\pi < 80\text{MHz}$ . Dotted line shows the behavior for an empty cavity. The inset shows an expanded view for  $|v-v_0|/2\pi < 100\text{kHz}$ .

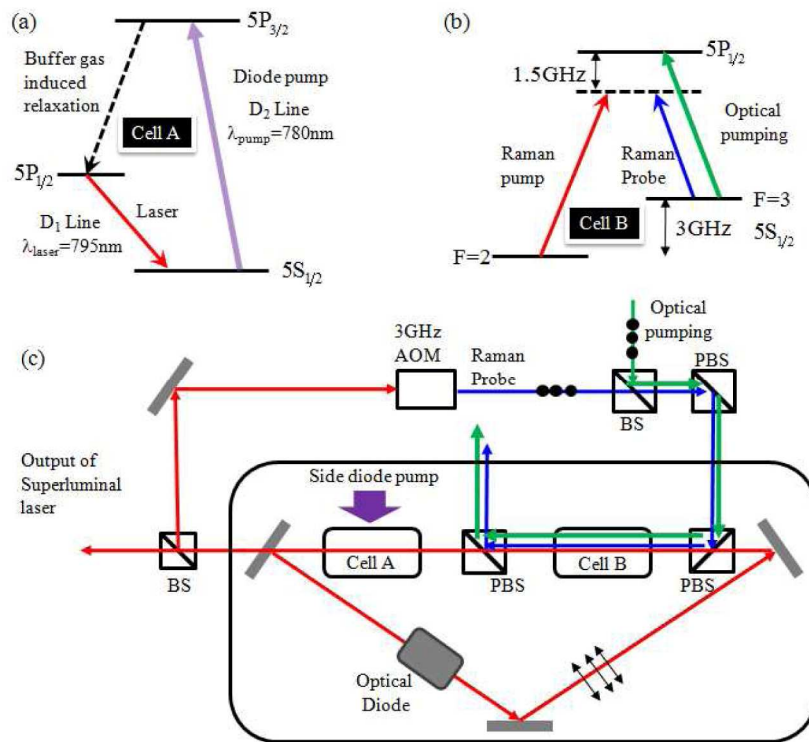


Fig. 4. Energy levels for (a) 795-nm Rb laser to produce broadband gain, (b) Raman depletion to induce narrowband absorption dip. (c) Schematics of the experimental set-up to realize a superluminal laser: PBS, polarizing beam splitter; BS, beam splitter; AOM, acousto-optic modulator. Note that the superluminal laser is the same as the Raman pump. The scheme shown is for  $^{85}\text{Rb}$  atoms. The broadband gain is produced by side-pumping with a diode laser array.

The scheme proposed here for a superluminal laser may be realizable in many different ways. For example, the gain profile can be produced by a diode pumped alkali laser [17]. The dip in the gain profile can be produced by inserting a Raman probe, following the technique shown in Ref.18. Figure 4 displays the corresponding energy levels, and the experimental configuration to realize a superluminal laser. Figure 4(a) illustrates the atomic transitions associated with the process for producing a broadband gain profile. The diode pump, applied from the side along the D2 transition, induces population inversion for the D1 transition in the

presence of a buffer gas of  $^4\text{He}$  [17]. Figure 4(b) shows the energy levels within the D1 transition used for producing the Raman depletion. The lasing beam acts as the Raman pump. The Raman probe is produced by sampling a part of the laser output, and shifting it in frequency with an acousto-optic modulator, operating at the frequency that matches the hyperfine splitting in the ground state [18]. An additional optical pumping beam is applied to produce the population inversion among the metastable hyperfine states necessary for the Raman gain and depletion. Thus, the laser light experiences the Raman dip in the cell B, as illustrated in fig.(c). Efforts are underway in our laboratory to realize such a scheme.

To summarize, we have shown that the effective dispersion experienced by a laser frequency is indirectly sensitive to the spectral profile of the unsaturated gain, even though the saturated gain becomes flat, equaling the cavity loss. Specifically, we have shown that a dip in the gain profile when properly tuned, leads to a superluminal group velocity for the lasing mode. The displacement sensitivity of the lasing frequency is enhanced by nearly five orders of magnitude with a wide range of sensing application, including gyroscopy, vibrometry and gravitational wave detection. We also outline an experimental scheme for realizing such a superluminal laser.

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