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The fundamental limitations on the practical realizations of white light cavities

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ABSTRACT

We study the possibility of realizing white light cavities (WLCs)—cavities which essentially resonate over a continuous band and not at discrete frequencies, by utilizing linearly chirped Bragg reflectors (LCBGs) for phase compensation. Analytical and numerical analyses show that this goal cannot be achieved because the reflection of a specific frequency by a LCBG cannot be modeled as occurring at the position where the Bragg condition is satisfied. The accumulated effect of multiple scatterings at different locations along the LCBG produces a positive group delay, preventing the realization of a WLC using this approach. We also present a generic, filter theory based, argument showing that any phase component that exhibit a negative group delay necessarily has a corresponding dip in its amplitude response. The implications of this conclusion on the limitations and design rules of WLC based devices are discussed in details.

1. Introduction

A white light cavity (WLC) is a unique type of resonator which is designed to resonate over a broad, continuous range of frequencies. This is in contrast to conventional cavities which resonate at discrete frequencies determined by their optical roundtrip length. The WLC concept exhibits unique properties which render it scientifically interesting as well as attractive for various applications. One of the most important properties of WLC is the elimination of the traditional relation between the linewidth of the cavity and its quality factor. In contrast to conventional cavities, a WLC possesses a broader linewidth than that of a conventional cavity with the same finesse [1,2]. In addition, the sensitivity of the lasing frequency of a WLC based laser to changes in the cavity length is substantially higher than that of conventional cavity based lasers. The combination of wide bandwidth on one hand and large finesse on the other is highly attractive for various applications, primarily in sensing and telecommunication. WLC based schemes have been proposed for enhanced gravitational waves detection [3], sensing [4,12] and for trap-door data buffering with a large delay-bandwidth product [5,6].

The key element in the realization of a WLC is a dispersive phase compensation mechanism having a negative phase slope with respect to the frequency (larger phase shift for lower frequencies) [12]. This mechanism equates the phase accumulated in a roundtrip by each frequency component, setting it to a multiple integer of 2π. It should be emphasized that the negative phase slope required for the realization of WLCs should not be confused with negative group velocity dispersion which is commonly employed for dispersion compensation in optical communications. Attaining the WLC conditions requires *negative group delay* (i.e. "superluminal" group velocity) and not negatively sloped group delay. Various approaches have been proposed and studied for the realization of such phase compensation mechanism, such as dual-pump Raman and Brillouin gain profiles [7–9], Brillouin absorption line [9], four wave mixing [10], and an intra-cavity resonator [4,11,12].

All of these approaches rely on the negative phase shift which is accompanied by a notch in the transmission spectrum due to the Kramers–Kronig (KK) relations, implying that attaining the WLC condition imposes additional constraints in the cavity that may be undesired [11,12]. Here, we explore an alternative approach employing linearly chirped Bragg gratings (LCBGs) as phase compensation mechanism. The intuitive rationale behind this approach is to exploit the fact that the reflection of different wavelengths effectively occurs at different positions along the gratings, and design gratings such that the cavity is effectively longer for lower frequencies. Consequently, the roundtrip phase accumulation would be frequency independent and a WLC would be formed. We find, however, that contrary to simple intuition, such LCBGs cannot be designed and that the accumulated effect of

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multiple scatterings at different locations along the LCBG leads to an overall positive group delay, thus impeding the formation of a WLC. The rest of the paper is arranged as follows. In Section 2 we present in details the proposed concept and particularly the design of LCBGs as frequency dependent reflectors. In Section 3 we present an unsuccessful attempt to realize a WLC utilizing LCBGs. In Section 4 we provide a filter-theory based argument which explains the outcome of Section 3 and outlines the limitation and necessary conditions for attaining a frequency dependent reflector which is appropriate for this task. In Section 5 we discuss the results and summarize.

2. WLC employing frequency-dependent reflectors

The main idea underlying the WLC concept is that the difference between the phases accumulated by different frequencies is compensated by a properly tailored, dispersive, intra-cavity medium or component. In such an arrangement, the roundtrip phase, ϕ, is frequency independent, i.e. $d\phi/d\omega=0$.

Consider a Fabry–Perot (FP) cavity of length L, which incorporate frequency dependent reflectors (FDRs): $r_j=|r_j|\exp(i\varnothing_{r_j})$ and $t_j=|t_j|\exp(i\varnothing_{t_j})$ ($j=1$, 2, see Fig. 1). In particular, we are interested in controlling the phase response of the reflectors in order to attain a WLC. The frequency response of the cavity is straightforwardly given by

$$E_{out}/E_{out}=\frac{|t_1t_2|exp(i\varnothing_{t_1})exp(i\varnothing_{t_2})exp(in_0L\omega/c)}{1-|r_1r_2|exp(i\varnothing_{r_1})exp(i\varnothing_{r_2})exp(2in_0L\omega/c)} \quad (1)$$

where n_0 is the index of the medium comprising the cavity and c is the speed of light in vacuum. Hence, the cavity power transmission $T_c=|E_{out}/E_{in}|^2$ is

$$T_C=\frac{|t_1t_2|^2}{1+|r_1r_2|^2-|r_1r_2|\cos(\Phi_{total})} \quad (2)$$

where $\Phi_{total}=\phi_{r_1}+\phi_{r_2}+2n_0\omega L/c$. Note that Φ_{total} is composed of three phase terms: the phases of the reflectors ϕ_{r_1} and ϕ_{r_2} and the roundtrip propagation phase in the cavity, $2n_0\omega L/c$. For simplicity, we assume that the two reflectors are identical, i.e. $\phi_r\equiv\phi_{r_1}=\phi_{r_2}$, $\phi_t\equiv\phi_{t_1}=\phi_{t_2}$, $|r|\equiv|r_1|=|r_2|$, and $|t|\equiv|t_1|=|t_2|$. Therefore, T_c can be rewritten as $T^2/[1+R^2-R\cos(\Phi_{total})]$ where $T=|t|^2$, $R=|r|^2$. A resonance frequency of the FP cavity, ω_0, must satisfy $\Phi_{total}(\omega_0)=2m\pi$ (m being an integer). The shift in the total phase accumulated by a frequency deviating from ω_0 by $\Delta\omega$ is given by

$$\begin{aligned}\Delta\Phi_{total}&=\Phi_{total}(\omega_0+\Delta\omega)-\Phi_{total}(\omega_0)\\&=2\left(\varphi_r(\omega_0+\Delta\omega)-\varphi_r(\omega_0)+\frac{n_0\Delta\omega L}{c}\right)\\&=2\frac{n_0\Delta\omega L}{c}\left(n_0+\frac{c\varphi_{rl}}{L}\right)\end{aligned} \quad (3)$$

where $\phi_{rl}=\partial\phi_r/\partial\omega|_{\omega=\omega_0}$. Eq. (3) employs a first order Taylor expansion of $\phi_r(\omega)$ around ω_0. The phase shift (Eq. (3)) can be canceled for every frequency if we design the reflectivity of the reflectors such that $\phi_{rl}=-n_0L/c$ over a certain bandwidth around ω_0. Ideally, such negatively sloped linear phase shift satisfies resonance condition for all frequencies (infinite bandwidth). In a realistic system, however, higher order dispersion of $\phi_r(\omega)$ would limit the WLC bandwidth.

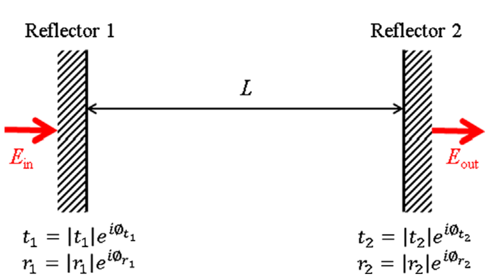

Fig. 1. Schematic illustration of FP cavity based WLC.

3. FDR implemented by linearly chirped Bragg grating (LCBG)

A natural candidate for the realization of FDRs is a pair of linearly chirped Bragg gratings (LCBG). The periodicity of the gratings of an LCBG is not constant but changes along the structure of the gratings. Because we are interested in compensating the larger phase shifts of higher frequencies, we need to have effectively a longer cavity for lower frequencies. A wave with wavelength λ_i is expected to be reflected primarily from the area in the reflector satisfying the Bragg condition, $d=(m+½)\lambda_i$ (d being the grating periodicity and m being an integer). Thus, the period of the LCGB should increase with the position of the effective reflector (see Fig. 2).

To analyze the Bragg reflection in LCBGs we define an effective optical path length Λ_i as the effective propagation distance from which a wave with wavelength λ_i is completely reflected. Λ_i can be viewed as the position of an effective metallic mirror which can replace the LCBG for wavelength λ_i, where it is desired to have larger Λ for longer wavelengths.

Because the grating period is linearly chirped, the Bragg condition can be written as $d(z)=(m+½)\lambda_i(z)$, where z is the optical axis, implying that different wavelengths are reflected from different positions in the LCBG. Thus, Λ_i, the periodicity of the grating corresponding to λ_i, increases with wavelength, as illustrated in Fig. 2(a), and therefore, the LCBG should be able to provide the necessary phase shift for the realization of a WLC. This conceptual approach is also supported by the results of Ref. [13]. Fig. 2(b) illustrates a FP cavity with length L which is formed by two LCBGs. The LCBGs are assumed to be identical and designed to be negatively chirped. To attain the WLC condition, the parameters of LCBGs such as chirping rate should be designed such that Λ_i satisfies $2\Lambda_i+L\equiv L_{eff}=m\lambda_i$ for all wavelengths λ_i.

To attain the reflectivity of a LCBG we write the electric field in the grating as a superposition of forward and backward-propagating waves with amplitudes $a(z)$ and $b(z)$, respectively (see Fig. 3). The complete field in the LCBG is given by $T_{total}(z)=a(z)e^{ikz}+b(z)e^{-ikz}$. The reflection coefficient at $z=-l$ is, therefore, $r=b(-l)/a(-l)e^{2ikl}$ where $k=n_0\omega/c$ is the wave number, n_0 is the average (unperturbed) index of LCBG which is assumed to be equal to that of the cavity. The index profile of the LCBG varies along the z-axis, such that $n(z)=n_0+\delta n(z)$. The perturbation δn is given by $\delta n(z)=2n_0\beta\cos[\theta(z)]$ where β is the modulation amplitude and $\theta(z)=\alpha z^2/2+\kappa z$ where κ is the modulation frequency at $z=0$ and α is

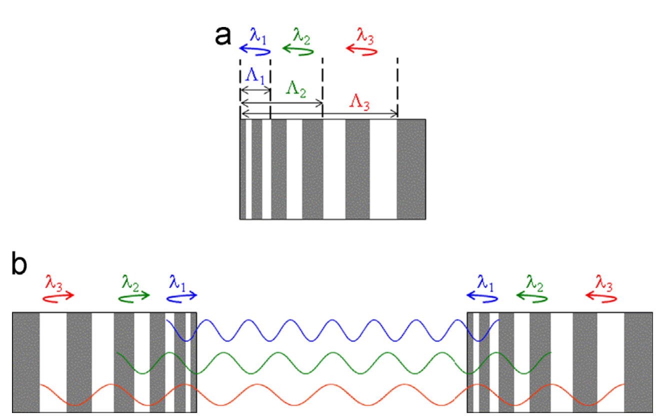

Fig. 2. LCBG based WLC realization: (a) three different wavelengths ($\lambda_1<\lambda_2<\lambda_3$) are reflected at different locations inside the grating region, (b) schematic illustration of a Fabry–Perot (FP) cavity of length L formed by a pair of LCBGs.

the chirping parameter. Note, that for $\alpha=0$, δn converges to the conventional, constant period, grating profile. Following [13], the reflectivity r is given by

$$r=\frac{\exp((1/2)i\alpha z_0^2)}{ik_0\beta}\frac{F(i\eta;(1/2);-(i\alpha/2)(l+z_0)^2)-\beta^2k_0^2\rho(l+z_0)F\left((1/2)+i\eta;(3/2);-(i\alpha/2)(l+z_0)^2\right)}{\rho F(-i\eta;(1/2);(i\alpha/2)(l+z_0)^2)-(l+z_0)F((1/2)-i\eta;(3/2);(i\alpha/2)(l+z_0)^2)}e^{i2kl} \quad (4a)$$

$$\rho=\frac{-F(i\eta;(1/2);-(i\alpha/2)(l-z_0)^2)}{\beta^2k_0^2(l-z_0)F((1/2)+i\eta;(3/2);-(i\alpha/2)(l-z_0)^2)} \quad (4b)$$

$$F(a;b;x)=\sum_{n=0}^{\infty}\frac{a_n}{b_n}\frac{x^n}{n!};\ a_0=\mathbf{1};\ b_0=\mathbf{1};\ a_n=a(a-\mathbf{1})\cdots(a-n+\mathbf{1});$$
$$b_n=b(b-\mathbf{1})\cdots(b-n+\mathbf{1}) \quad (4c)$$

where $z_0=(2k-\kappa)/\alpha\equiv 2q/\alpha$, $\eta=\beta^2k_0{}^2/(2\alpha)$ and $k_0=n_0\omega_0/c\equiv\kappa/2$ for a first order Bragg grating. The region of the LCBGs is assumed to be $-l\le z\le l$. z_0 can be interpreted as the effective reflection point of the field with the wavenumber k [12] and is related to the effective optical path length $\Lambda(\lambda)$, defined in Fig. 2, as $\Lambda=z_0+l$. Clearly, Λ is frequency dependent and the LCBG provides a frequency dependent phase shift. The reflection phase of the LCBG is given by $\phi_r=\phi_{LCBG}+2kl$ and the total reflectivity is $r=exp(i\phi_{LCBG}+2ikl)$ [15].

To attain a WLC, the phase accumulated due to conventional propagation must be compensated by the LCBGs. It is constructive to represent the phase shift of the LCBG as being equivalent to the phase accumulated due to propagation through a medium of length $2l$ with an effective index, n_{eff}, such that $\phi_r=2ln_{eff}\omega/c$. Note, that n_{eff} is frequency dependent. The corresponding group index is given by $n_g=n_{eff}+\omega\partial n_{eff}/\partial\omega=c/2l\phi_{rl}$ [21]. Thus, to obtain the WLC condition this group index must satisfy $n_g=-L/2ln_0$ (for two identical LCBGs).

The next step is to identify a parameter space in which the LCBGs can satisfy the WLC condition, i.e. $\phi_{r1}=-n_0L/c$. Fig. 4 depicts an example for the reflectivity and the group delay for both negatively and positively chirped grating for $n_0=1.45$,

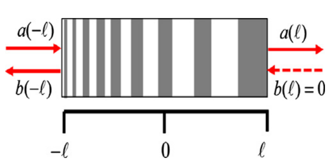

Fig. 3. Field propagates along $z=-l\sim l$. $a(-l)$ and $a(l)$ are the input and the transmitted field, respectively. $b(-l)$ is the reflected field.

$l=5\times10^{-4}$ m, $\beta_0=6.7\times10^{-4}$, $k_0=6\times10^6$ rad/m and $\alpha=\pm6\times10^6$ rad/m^2. The absolute value of the reflectivity $|r|^2$ is identical in both chirping cases (see Fig. 4(a)). The group delays, on the other hand, are mirror images of each other, which is in agreement with the results of Ref. [13]. Note that $\partial\phi_r/\partial\omega>0$, i.e. positive group delay is attained in the high reflectivity regime corresponding to $|r|^2\approx 1$, for both negatively and positively chirped grating. Moreover, the obtained group delay in this regime is almost identical for both positive and negative chirpings. In order to identify a negative group delay regime, a thorough numerical search has been conducted over the parameter space of LCBGs (modulation depth, chirping rate and grating length).

Fig. 5 depicts the *minimal* group delay in the LCBG reflection band as a function of the chirping parameter α and the grating modulation depth β for a given grating length $2l$. Group delay maps calculated for both longer and shorter grating lengths were practically identical to that shown in Fig. 5. As shown in the figure, the attained group delay is positive for all the values of the parameters in the scan range. The group delay does decrease as the modulation depth is increased but remains positive.

To verify (4a)–(4c) as well as the results presented Fig. 5, we have calculated numerically the reflectivity of the LCBG using a transfer matrix method (TMM) [15,16]. For the calculation, we model the gratings by many piecewise constant sections which are much smaller than the grating period. In each section, the field is a superposition of forward and backwards propagating plane waves. By applying the appropriate boundary conditions at the interface of successive sections we can get a relation between the fields in these sections in the form of a transfer matrix. From the product of these matrices it is possible to extract the reflectivity of the structure at each frequency. Fig. 6 depicts the reflectivity and group delay obtained by the TMM. For the calculation, the LCBG structure is divided into $64N$ sections (N being the nearest integer to ln_0/λ_0) corresponding to the optical path length due to each section where $\lambda_0=2\pi c/\omega_0$. The results of the TMM analysis were found to be identical to those of the analytical approach presented in Fig. 4 and at the high reflectivity band ($|q|\le 4000$) the group delay is indeed found to be positive.

The difficulties in obtaining a negative group delay using LCBGs, in contrast to Ref. [13], are rather unexpected. Although it would be naïve to expect the position of the "effective" mirror for λ_i

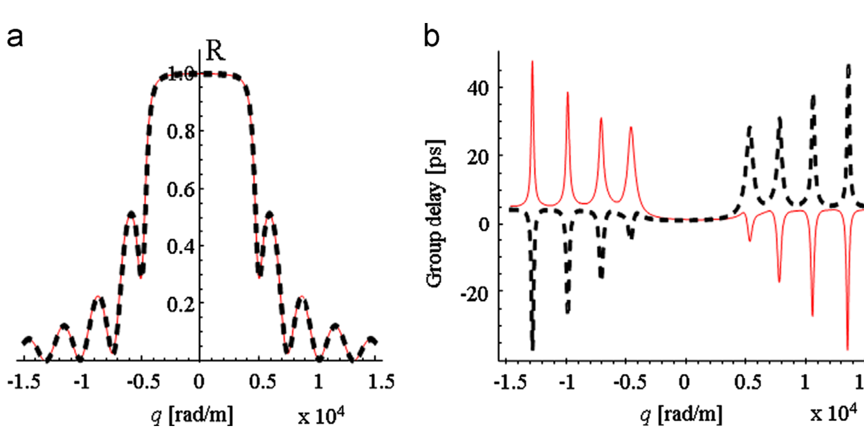

Fig. 4. Reflection (a) and group delay (b) of LCBG calculated according to Eq. (4a). Red solid line: $\alpha<0$, dashed line: $\alpha>0$. The parameters are defined in the text. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

to be at the point where the gratings periodicity satisfies the Bragg condition for that wavelength, it is reasonable to assume that negative chirping would be manifested by some level of negative group delay. The fact that the resulting group delay is actually positive indicates that the chirping of the grating is not necessarily the dominant mechanism in the determination of the effective reflection point, i.e. the position of the "effective" mirror. Fig. 7 illustrates the position of this "effective" mirror (calculated according to the group delay) as a function of the wavelength for the positively and negatively chirped grating of Figs. 6 and 4. For comparison, the effective mirror position for non-chirped gratings (but with identical index modulation depth) is overlaid on the figure. Clearly, within the reflector bandwidth the position of the "effective" mirror is essentially more remote for shorter wavelengths, even for the negatively chirped grating, which is in contrast to what might be expected based on the chirping profile. It seems that the chirping merely introduces a bias on the effective position of the mirror which is negative (positive) for negative (positive) chirp parameter α. The magnitude of the bias is determined by the magnitude of the chirping parameter α; where larger α yields larger bias.

The dependence of the effective mirror position on the wavelength (Fig. 7) indicates that the underlying idea of utilizing negatively chirped grating for achieving effectively longer cavities for longer wavelengths does not work. Despite the negative chirp, the LCBG generates longer cavities for shorter wavelength, thus increasing the phase difference of normal propagation instead of compensating it. The primary difference between positively and negatively chirped gratings is essentially an overall bias of the effective mirror position, with respect to non-chirped gratings, were the sign of the bias depends on the sign of the gratings.

This outcome stems from the non-zero bandwidth of Bragg reflection originating from the finite modulation depth of the refractive index. The reflection bands of the apparently distinct reflectors composing the LCBR overlap, thus impairing the intuitively expected negative phase profile. Finally, we would like to point out that the incorrect result in ref [13] stems from an inadvertent error in defining phase shifts of the optical fields as functions of coordinates of the grating [14].

4. Is there a fundamental problem?—a filter theory approach

The failure to attain a negatively sloped phase profile from the LCBG raises the question whether there is a fundamental problem or we have simply made an unfortunate choice of parameter space. Negative slope phase profiles have been demonstrated previously by taking advantage of the KK relations [7,9], which are the relationships between the real and imaginary parts of the refractive index. Unfortunately, achieving negative slope phase profile using this approach requires a dip in the amplitude transfer function which reduces the Q-factor of the (white light) cavity and its bandwidth. This is in contrast to the LCBG which intuitively should provide a flat amplitude transfer function and a negatively sloped phase response.

It is convenient to consider the LCBG as a causal and stable linear system (or filter) which operates on an optical signal and, thus, can be analyzed by the tools of signal processing theory. Linear systems theory has its own counterpart to the KK relation in the form of the Hilbert transform. The phase response of a causal and stable system can be uniquely extracted from its amplitude response if the system has minimal phase, i.e. if the zeros of the

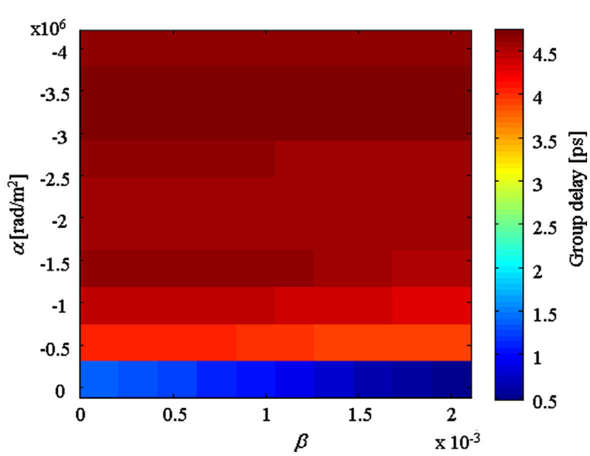

Fig. 5. Minimal group delay in the LCBG reflection band. The grating length is 2 mm and the center wavelength is 1.52 μm.

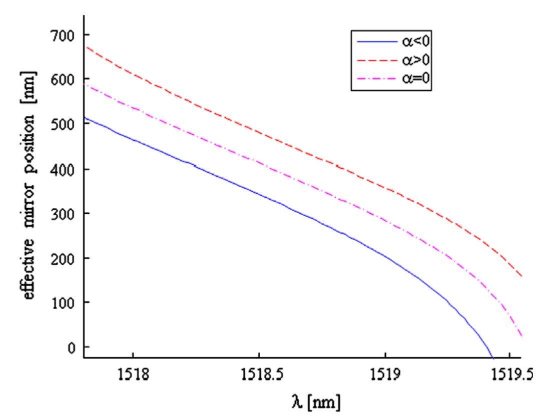

Fig. 7. Effective mirror position with respect to the grating edge within the reflection bandwidth of the LCBGs. Dash-dot magenta—no chirp, dashed red—positive chirp, solid blue—negative chirp. The gratings parameters are as in Figs. 4 and 6. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

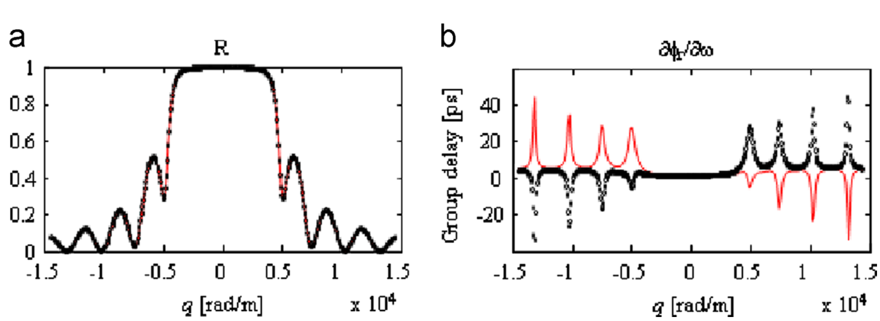

Fig. 6. Reflection (a) and group delay of LCBG calculated by TMM. Red solid line: $\alpha<0$, dashed line: $\alpha>0$. The parameters are identical to those used for Fig. 4. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

transfer function $H(\omega)$ of the system are within the unit circle [17,18]. This relation is given by

$$H(\omega)=\exp[\alpha(\omega)+i\phi(\omega)]$$
$$\phi(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha(\omega-\omega')}{\omega'}d\omega' \tag{5}$$

Note that for a system to be causal and stable, it is not required to satisfy this additional constraint. In such a system, however, the phase response cannot be uniquely determined by the amplitude response.

A straightforward example for such system is the all-pass filter which can be implemented in optics by the Gires–Tournois interferometer. This interferometer is similar to the well-known Fabry–Perot interferometer except that one of the mirrors has 100% reflectivity. As a result, the magnitude of the reflectivity of the interferometer is unity at all frequencies but the phase response is *not* constant as might be expected by applying the KK relations or the Hilbert transform. The reason for that is that the all-pass filter is not a minimum phase filter because its zeros are not on the left hand side of the complex *S* plane.

An important property of minimum phase filter (or any minimum phase linear system) is that it also exhibits the minimal possible group delay, i.e. for a given amplitude response, the system which exhibits the minimal group delay is the minimum phase system and, therefore, satisfies the KK relations. This property has a direct impact on the attempt to realize a transfer function combining negative slope phase profile and flat, close to unity, amplitude profile.

For simplicity, we restrict the analysis to systems with amplitude responses that are symmetric around a central frequency ω_0. Let us assume that we have a reflector with a flat top amplitude response (like the LCBG for example). The group delay of the minimal phase filter, possessing the minimal group delay, is proportional to the derivative of $\phi(\omega)$ with respect to ω. A well-known property of the Hilbert transform is that the derivative of the transform of a function equals the transform of the derivative of the function

$$\frac{d\phi}{d\omega}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha'(\omega-\omega')}{\omega'}d\omega'=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha'(\omega')}{\omega+\omega'}d\omega' \tag{6}$$

where α' is the derivative of α with respect to ω. Let us consider a generic, minimal phase, band-pass filter centered at a frequency ω_0 and bandwidth $\Delta\omega$, as illustrated in Fig. 8, and use Eq. (6) to extract the properties of its phase response. Note that Fig. 8 depicts the logarithm of the amplitude response, i.e. α in Eq. (5). In addition, because we consider a generic bandpass filter, we note that α' is monotonically decreasing, i.e. $\alpha''<0$ for all ω. As the Hilbert transform of a shifted function is shifted as well

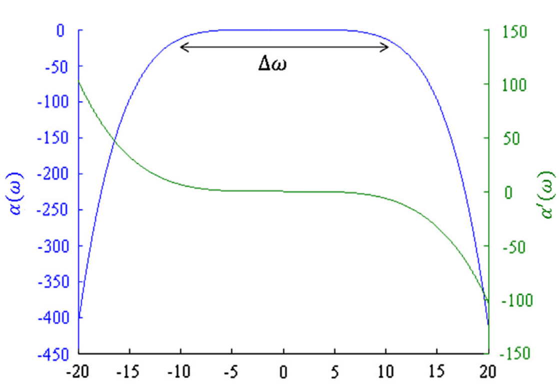

Fig. 8. α (blue) and the corresponding α' (green) of a generic minimal phase bandpass filter centered at $\omega=0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

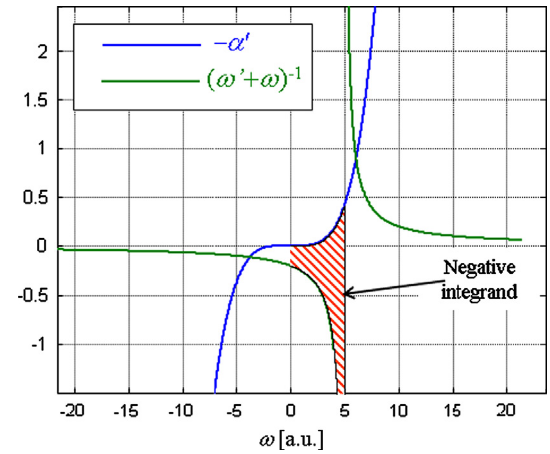

Fig. 9. The negative contribution to the convolution integral (Eq. (6)).

$[\mathcal{H}(\alpha(\omega))=\varnothing(\omega)\Rightarrow\mathcal{H}(\alpha(\omega-\omega_0))=\varnothing(\omega-\omega_0))]$, we can consider the filter as centered at $\omega_0=0$. Fig. 8 also illustrates the derivative of α which is positive for $\omega<0$ and negative for $\omega>0$. The calculation of the group delay (Eq. (6)) is essentially a convolution of α' with $1/\omega$, which integrates over the shifted and frequency inverted α' multiplied by $1/\omega$. Note that the result of Eq. (6) is positive for all values of ω. At $\omega=0$ (i.e. at the center of the filter), the integrand is a product of two anti-symmetric function which is positive at all ω', thus yielding a positive group delay.

Let us now consider the case of $\omega<0$. As illustrated in Fig. 9, a section of the positive part of the frequency inverted α' is multiplied by a negative section of $1/\omega'$, thus yielding a negative contribution to the integral (this section is marked in Fig. 9). Note that in this case it is not obvious that the overall integral yields a positive contribution. To verify that, we separate Eq. (6) to its positive and negative parts

$$-\int_{-\infty}^{\infty}\frac{\alpha'(\omega')}{\omega+\omega'}d\omega'=-\left\{\int_{-\infty}^{0}\frac{\alpha'(\omega')}{\omega+\omega'}d\omega'+\int_{|\omega|}^{\infty}\frac{\alpha'(\omega')}{\omega+\omega'}d\omega'+\int_{0}^{|\omega|}\frac{\alpha'(\omega')}{\omega+\omega'}d\omega'\right\} \tag{7}$$

where the last term in the RHS of Eq. (7) is the negative contribution. In order for Eq. (7) to be positive (as we claim), the first two terms in the RHS of Eq. (7) must be larger than the absolute value of the last term (keeping in mind that ω is negative). To show that Eq. (7) is indeed positive for any ω, we consider a part of the integral Eq. (7)

$$I=-\int_{0}^{|\omega|}\frac{\alpha'(\omega')}{\omega'-|\omega|}d\omega'-\int_{|\omega|}^{2|\omega|}\frac{\alpha(\omega')}{\omega'-|\omega|}d\omega' \tag{8}$$

Note that ω is assumed to be negative. Introducing $\omega''=\omega'-|\omega|$ and changing the integration boundaries yields

$$I=-\int_{-|\omega|}^{0}\frac{\alpha'(\omega''+|\omega|)}{\omega''}d\omega''-\int_{0}^{|\omega|}\frac{\alpha'(\omega''+|\omega|)}{\omega''}d\omega''$$
$$=\int_{0}^{|\omega|}\frac{-\alpha'(\omega''+|\omega|)+\alpha'(|\omega|-\omega'')}{\omega''}d\omega' \tag{9}$$

Because $-\alpha'$ is a monotonically increasing function, *I* is necessarily positive and therefore Eq. (7), which includes only positive contributions in addition to *I*, is also essentially positive.

A similar argument applies for positive values of ω. We can, therefore, conclude that the group delay of such minimal phase filter is essentially positive. Fig. 10 depicts a plot of the Hilbert transform (or phase response) of α corresponding to a bandpass filter and the resulting group delay τ_d. The apparently negative group delay close to edges of the frequency window is a numerical

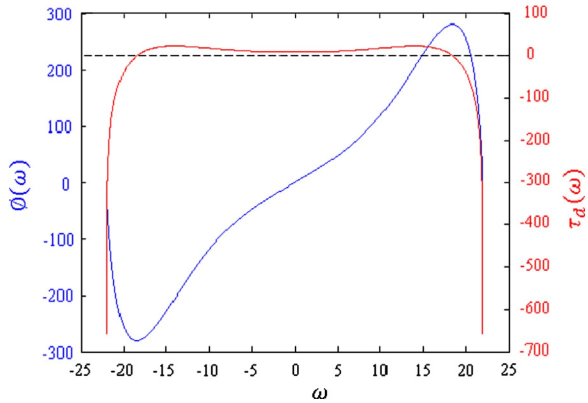

Fig. 10. Phase response (blue) and the corresponding group delay (red) of a generic minimal phase bandpass filter centered at $\omega=0$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

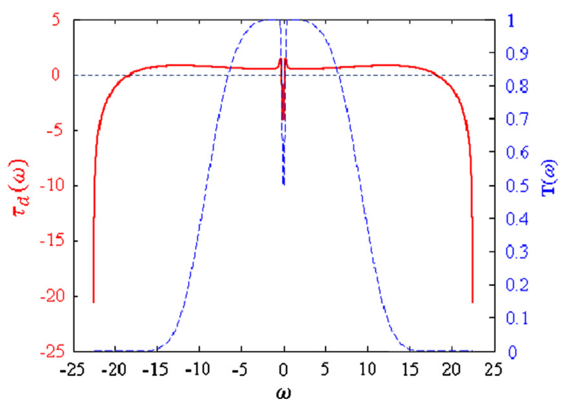

Fig. 11. Group delay (red) and amplitude transfer function (blue) of a flat-top filter incorporating a notch at the center of its pass-band. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

artifact stemming from the finite frequency range used for calculating Eq. (6) (see also Appendix A).

The important outcome of the argument above is that the group delay associated with a minimal phase bandpass filter is positive. However, any other bandpass filter must possess larger group delay than that of the minimal phase filter, and therefore must have a positive group delay as well. The minimal attainable group delay from such a filter is zero, which can be achieved for a filter having a completely flat response over the whole spectrum, i.e. a filter having $\alpha'=0$. An example for such minimal phase filter is an ideal perfect metallic mirror. Thus, it is *impossible* to realize a causal and stable flat top reflector which provides a negative group delay, which explains our failure to obtain such a filter using LCBGs.

In order to attain a negative group delay, for example at $\omega=0$, it is necessary to get α' to obtain negative values at least for some region at $\omega<0$ and positive values at $\omega>0$, which essentially generates a dip in the transmission function. To illustrate this point we plot in Fig. 11 the amplitude transfer function and the corresponding group delay of a filter similar to that depicted in Fig. 10 where a narrow notch was introduced at the center of the pass-band. A negative group velocity region is indeed formed around the deepest point of that notch. Note that a peak at $\omega=0$ would increase the group delay making it even more positive. It is important to understand that the existence of a notch in the transfer function is a *necessary* but not *sufficient* condition for attaining negative group delay. Such a notch must provide sufficient negative contribution to Eq. (6) to overcome the positive part of the integral. Thus, the outcome of this analysis is of fundamental significance as it implies that any realization of a white light cavity requires an element having a dip in its amplitude response. This, in turn, introduces an inherent limit on the actual achievable bandwidth of any WLC.

5. Discussion and summary

The important outcome of the research presented here is that any realization of a negative group delay component poses some constraints on its amplitude response. For example, using filter theory based argument we show that a negative group delay device cannot possess a flat amplitude response—an outcome which we have demonstrated for a specific implementation utilizing LCBGs. Instead, it is essential to have a notch in the amplitude response of such a component in order to attain a negative group delay.

This fundamental result introduces significant implications to WLC based devices and systems, especially when considering the structure of the negative group delay component (e.g. the LCBG studied here). Generally speaking, there are two primary and generic approaches for achieving an amplitude transfer function exhibiting a “notch”. The first approach includes the incorporation of an element with an effective absorption line. A straightforward implementation of such a component consists of two Bragg reflectors centered at different wavelength but with overlapping slopes. The main drawback of this type of implementation is that it introduces inherent loss into the phase components and consequently reduces the Q-factor and the bandwidth of the WLC in which it is incorporated.

The second generic approach to attain the necessary amplitude response (and hence, the desired phase response) is by introducing a dual peak gain profile as demonstrated in [19,20]. Negative group delay can be formed in the notch located between the gain peaks and facilitates the satisfaction of the WLC condition within that spectral range. Compared to the passive (loss-only based) approach, the incorporation of gain seems to overcome the main drawback which is often associated with negative group delays—loss, thus allowing for the realization of WLCs having high Q-factors. However, the incorporation of gain has its own drawbacks, in particular the additional noise which is inherent to any gain mechanism. Depending on the application, additional noise can be a significant problem. Furthermore, it is in general difficult to realize a WLC with sufficient bandwidth for applications such as data buffering [7] while suppressing lasing of adjacent longitudinal modes of the cavity.

In conclusion, we studied the possibility of using a frequency dependent phase implemented by LCBG in order to realize a WLC. Contrary to the straightforward intuition, we were unable to find a set of design parameters for which the desired phase response (i.e. negative group delay) can be attained. Although a negative phase slope can be attained at a discrete set of frequencies, the overall (continues) group delay, even in this case, remains positive. Based on control and filter theory we have presented an argument showing that it is fundamentally impossible to attain a reflector (or any linear and causal system) exhibiting simultaneously a flat amplitude frequency response and a negative group delay. This conclusion is of significance because it implies that any realization of a white light cavity inherently requires an element having a non-uniform amplitude transfer function which contains a dip. This is a fundamental result which may introduce important constraints on the utilization of WLC based components in optical devices.

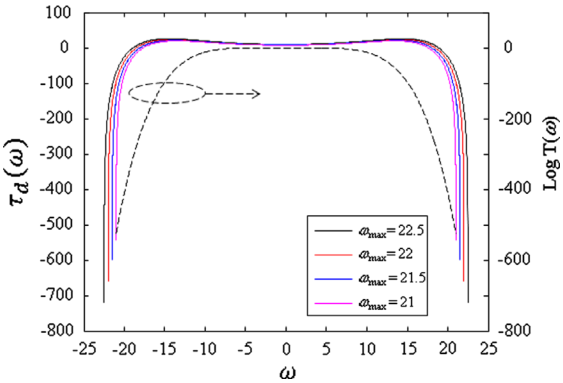

Fig. A1. Dependence of the group delay calculated by the Hilbert transform of the frequency calculation window. Dashed black—log plot of the amplitude transfer function of the filter.

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Appendix A

The theoretical argument presented in Section 4 shows that the group delay associated with a minimum phase and smooth bandpass filter (exhibiting no local transmission dips) should be positive for all frequencies. Nevertheless, the group delay of such filter, numerically calculated by the filbert transform exhibits negative group delays outside the pass band (see Fig. 10). We believe these spurious negative delays stem from numerical artifact caused by the finite frequency range used for the calculation. To verify that, we plot in Fig. A1 the group delays of the filter calculated according to Eq. (6) for various frequency calculation windows. A log scale plot of the filter transmission is superimposed for comparison reasons. Clearly, as the calculation window is *decreased* (corresponding to *less accurate* group delay calculation), the negative group delay regions are pushed towards the pass-band of the filter. Moreover, it should be emphasized that the filter amplitude transfer level at these spurious negative group delay regions drops below 10^{-130}, thus rendering the accuracy of the group delay calculation in these regions quite problematic.

References

[1] R.H. Rinkleff, A. Wicht, Physica Scripta T118 (2005) 85.
[2] A. Rocco, A. Wicht, R.-H.Rinkleff, K. Danzmann, Physical Review A 66 (2002) 053804.
[3] A. Wicht, K. Danzmann, M. Fleischhauer, M. Scully, G. Miller, R.H. Rinkleff, Optics Communications 134 (1–6) (1997) 431.
[4] M. Salit, G.S. Pati, K. Salit, M.S. Shahriar, Journal of Modern Optics 54 (16–17) (2007) 2425 10–20 November.
[5] J. Schaar, H. Yum, M.S. Shahriar, Practical considerations for a fast-light enhanced helium-neon ring laser gyroscope, in: Proceedings of the SPIE Photonics West, 2011, San Francisco, CA.
[6] H.N. Yum, M.E. Kim, Y.J. Jang, M.S. Shahriar, Optics Express 19 (7) (2011) 6705.
[7] H.N. Yum, X. Liu, Y. Jang, M. Kim, M.S. Shahriar, Journal of Lightwave Technology 29 (2011) 2698.
[8] G.S. Pati, M. Salit, K. Salit, M.S. Shahriar, Physical Review Letters 99 (2007) 133601.
[9] H.N. Yum, M. Salit, P.R. Hemmer, M.S. Shahriar, Demonstration of White Light Cavity Effect Using Stimulated Brillouin Scattering in a Fiber Loop, preprint at http://lapt.ece.northwestern.edu/preprints/wlc-demo-in-fiber.pdf, 2013, In preparation.
[10] H.N. Yum, M. Salit, J. Yablon, K. Salit, Y. Wang, M.S. Shahriar, Optics Express 18 (17) (2010) 17658.
[11] RobertFleischhaker, Jörg Evers, Physical Review A 78 (2008) 051802(R).
[12] O. Kotlicki, J. Scheuer, M.S. Shahriar, Optics Express 20 (2012) 28234.
[13] O.V. Belai, E.V. Podivilov, D.A. Shapiro, Optics Communications 266 (2006) 512.
[14] D.A. Shapiro, Personal Communications.
[15] Note that the phase of the reflection coefficient appearing in Eq. (15) of Ref. [13] does not include the phase term.
[16] P. Yeh, Optical Waves in Layered Media, Wiley, 1991 Chap. 6 pp. 155–205.
[17] T. Kailath, A.H. Sayed, B. Hassibi, Linear Estimation, 1 edition, Prentice Hall, (2000) p. 235.
[18] G. Lenz, C.K. Madsen, Journal of Lightwave Technology 17 (1999) 1248.
[19] L.J. Wang, A. Kuzmich, A. Dogariu, Nature 406 (2000) 277.
[20] G.S. Pati, M. Salit, K. Salit, M.S. Shahriar, Physical Review Letters 99 (2007) 133601.
[21] J. Scheuer, Europhysics Letters 77 (2007) 44004.