Field of Electrical Engineering

By

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ABSTRACT

Polarization Imaging System Using a Volume-Grating Stokes-meter

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We have developed a polarization imaging system using thick multiplexed holograms to identify all four Stokes parameters of the input beam. We derive the Mueller matrix defined in terms of diffracted amplitudes of planar and perpendicular polarization components, and determine the constraints for reconstructing the complete Stokes vector. Highly polarization-sensitive holographic gratings required for a
holographic Stokesmeter (HSM) has been made. These gratings can be accurately described by coupled-wave analysis. A numerical analysis of the noise tolerance of the Stokesmeter has been performed, and the gratings demonstrated allow the construction of an HSM. The use of a heterodyne receiver architecture can lead to additional gains in the signal-to-noise ratio.

We then have performed measurements of arbitrary Stokes parameters required for a polarimetric imaging with the HSM that consists of two sets of rotated orthogonal gratings and a quarter-wave plate. These measured values are compared well to values measured using the quarter-wave plate/linear polarizer method, which establishes the feasibility of such a Stokesmeter in its original configuration. We demonstrate further the basic mechanism behind a compact architecture for this device, requiring only a single substrate and a single imaging system, and describe a spectrally scanned polarimetric imaging system. We have demonstrated this spectrally scanned HSM by measuring arbitrary Stokes parameters of input beams for two different wavelengths. The ability to combine spectral discrimination with polarization imaging makes this HSM a unique device of significant interest.

Photonic Band Gap (PBG) materials with spatially periodic dielectric functions can be considered as a limit of volume holographic mirrors with a high-contrast periodic modulation of their complex refractive index. We briefly describe general properties of PBG materials and show some fabrication examples to implement 2D PBG structures. We have shown the photonic band gaps as well as defect modes (such as a cavity mode and a waveguide mode) of photonic crystals using Finite-Difference Time-Domain method. We have further demonstrated a polarization-dependent photonic band gap of a
2D photonic crystal based on this FDTD approach, which allows one to construct a new type of volume-grating Stokesmeter. In addition, we describe a spectrally resolved polarimeter using the scaling property of photonic band gap materials.

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Chapter 1

Introduction

Polarization is a property of electromagnetic radiation describing the shape and the orientation of the locus of the electric field vector extremity as a function of time, at a given point of the space. When light is passed through a medium, or is reflected by a target, its polarization state is transformed. For example, Figure 1.1 shows a polarized filter used in sunglasses and photography to reduce unwanted glare. Depending on the illumination angles, scattered skylight can have a significant level of polarization. Therefore, the variations in the polarization state of light enable one to characterize the system under consideration.

Polarimetric imaging [1-4] takes advantage of the fact that a given object emits and scatters light in a unique way depending on its polarimetric signature. This polarimetric signature is dependent upon several characteristics of the target including the composition and the roughness of the target surface, the shape and orientation of the target. Identifying the components of the polarization of the light reflected from the target

(a) Photograph of a window using (a) a vertically polarized filter (b) a horizontally polarized filter

Figure 1.1

- 13 -
allows one to construct an image that corresponds to the target’s unique polarimetric signature [5]. Such an image can reveal contrasts between two zones of the scene that have the same intensity reflectivity (so that no contrast appears in the intensity imaging) but different polarimetric properties, which can improve the detectability of small, low contrast objects in images. This polarimetric imaging is useful in applications such as target recognition, vegetation mapping, pollution monitoring, geological surveys, medical diagnostics, computer vision, and scene discrimination [6-10].

A typical example of a polarimetric imaging system is shown in Figure 1.2 [11]. The aluminum disk in Figure 1.2(a) has two scratched patches. The left patch is scratched vertically while the right patch is scratched horizontally. Except for the two patches, the disk face is sandblasted. The scratched patches have facets that act like mirrors at different orientation angles. These facets selectively reflected light incident on the object from specific direction toward the camera, while the reflected light from the sandblasted face is nearly unpolarized under the illumination. Conventional intensity imaging system creates images by measuring and recording the signal intensity as a function of position within a frame. This conventional imaging system is polarization-blind imaging system that measure intensity only as in Figure 1.2(b), so that it doesn’t represent these two patches. On the other hand, the polarization image in Figure 1.2(c) can discriminate these
two patches clearly because the scattered or reflected lights from the object have different polarimetric signature depending on the spatial location of the object.

Figure 1.3 also illustrates the advantage of the polarization imaging technology. Intensity imaging sensor (LEFT) have a difficult time identifying the actual target from intense plume and high background (sky) intensities. An active system (RIGHT) that emits light with a desired polarization (red solid circles) and only images backscattered light of a given polarization (blue dashed circles) ignore both high intensity source(sky and rocket plume), creating a high-contrast image of the desired target.

The aim of this thesis is to develop the volume-holographic Stokesmeter (HSM) for a polarimetric imaging system that has many advantages over the current methods. To accomplish this, in chapter 2, we briefly review the current methods for polarization imaging, and mention the advantages of the HSM. In chapter 3 we review the basic concepts of Stokes parameters and Mueller matrix needed to implement the polarimetric imaging system, and provide some background into holographic theory and our material selection. In chapter 4, we present theoretical analysis for the holographic grating using
the Mueller matrix defined in terms of diffracted amplitudes of planar and perpendicular polarization components, and then show these holographic gratings are described by coupled wave theory. After arming ourselves with the appropriate theoretical tool, in chapter 5, we demonstrate polarization sensitive holographic gratings, and suggest a heterodyne architecture to get additional gains in signal-to-noise ratio. In chapter 6, we present a compact version of this HSM with a single imaging system and two electro-optic modulators. In addition, we show the ability to combine spectral discrimination with polarization imaging by demonstrating the spectrally scanned HSM.

A photonic band gap (PBG) material is a special type of hologram with a high contrast ratio of dielectric constants in a volume grating. Since an inherent polarization sensitivity of a volume-grating, in general, can be used to determine all four Stokes parameters, we can implement a volume-grating Stokesmeter using a PBG material. Accordingly, in chapter 7, we present general properties of a PBG structure and FDTD simulation methods, and then illustrate some fabrication examples to implement a photonic crystal. In chapter 8, we demonstrate photonic band gaps and localized defect modes using Finite-Difference Time-Domain simulation. We further describe a volume-grating Stokesmeter using a polarization-dependent photonic band gap, and then present a spectrally resolved Stokesmeter based on the scaling property of a photonic crystal. Finally, in the last chapter 9, we have summarized the work for this thesis, and present proposals for future work.
Chapter 2

Polarization Imaging

2.1 Current Methods

Identifying the Stokes vector of the scattered light completely characterizes the polarization of the light [5,12,13]. Current architectures for such a Stokesmeter include mechanical quarter wave plate/linear polarizer systems, photo-detectors with polarization filtering gratings etched directly onto the pixel, and liquid crystal variable retarders [14-18].

Polarization filters etched onto pixels provide some images with polarization filtering but cannot resolve the full Stokes vector. Mechanical wave plate/polarizer combinations are the traditional method of determining the incident Stokes vector, and is the principle method employed in ellipsometry measurements. In order to record the full Stokes vector the retarders and linear polarizers must be precisely oriented, an image is collected, and then the retarder/linear polarizer combination must be re-oriented. As a result this system records individual Stokes parameters sequentially. Consequently, the speed of the system is very limited by the precise rotation of the optical components that is required and does not lend itself to a compact architecture. Liquid crystal variable retardation is a very similar to the rotating polarizer system where the individual retarders and polarizers are replaced by a liquid crystal display. While the liquid crystal display is more easily controlled than a mechanical system, it is still a sequential process with significant time required between captures. The system’s overall speed is limited by the
time it takes for the liquid crystal display to re-orient itself, which is on the order of 100 milliseconds per scan. This limits the throughput to ~ 10 Hz. The sequential nature of both approaches allows the scene to change between image captures – particularly in high-speed intercept scenarios. Further, LCD based systems also suffer from spectral selectivity, as the phase retarding function has limited range.

Systems based on the wave plate/polarizer architecture that split the beam into four parts and calculate the parameters in parallel require a large number of optical components [19]. Grating based polarimeter uses multiply diffracted and dispersed orders, and can measure all four Stokes parameters in parallel, but always needs the order blocking filters to prevent overlapping between the multiple diffraction orders of the beam [20]. Use of these filters leads to unwanted scattering, and may not be suitable for polarimetric imaging.

### 2.2 Holographic Stokesmeter

The holographic Stokesmeter (HSM) can be used to resolve all four Stokes components in parallel and at a high speed [21]. For a typical thick holographic grating, the response time of the device can be on the order of 10 ps. This removes the device as the bottleneck in the imaging system and instead the desired signal-to-noise ratio and detector parameters set the upper limit on the imaging process.

The design utilizes the polarization sensitivity of diffraction gratings in a thick hologram. Multiple gratings and a quarter wave plate are used to determine the Stokes vector parameters of incident light, producing a polarization sensitive, rather than an
intensity-only sensitive sensor. Our design for a polarimetric imaging system is presented in Figure 2.1. In the figure, the collected image enters the system at left, is focused, and then split into two copies. One copy is diffracted into two beams by two holographic gratings. The other copy is passed through a quarter-wave plate (QWP) and then also diffracted. Lenses or mirrors may be applied to the problem of image focusing. Mirrors

\[
\begin{align*}
A^* + B^* + C^* + D^* &= \\
\end{align*}
\]

*Figure 2.1 Schematic of Holographic Polarimetric Imaging Sensor (HPSI). Top panel: Incident light (image) enters system at left, is focused and split into two copies using a beam splitter. Each image is then directed at the holographic diffractive element (HDE). Each incident beam is diffracted into two beams, and the four, total, diffracted beams are projected onto four CCD arrays where the intensities are recorded. Bottom panel: Intensities on the CCD Arrays are multiplied by pre-determined coefficients and summed to image only the desired polarization, or Stokes vector.*

are commonly used in multi spectral applications to minimize chromatic aberrations. The diffracted beams containing the images are projected onto CCD arrays. The four intensities of the CCD arrays are summed with a pre-determined weighting in order to
image each component of the Stokes vector. This design takes advantage of the fact that a holographic grating is sensitive to the polarization state of the incident light. By adding and subtracting diffracted intensities, the incident Stokes vector can be identified and imaged. An advantage of our design is that the desired polarimetric signature can be changed continuously. This is because the specific polarimetric signature being imaged is determined by appropriate summing of the four images, which can be varied electronically. Also, since the hologram separates the intensities simultaneously, the holographic element response is on the order of a few pico-seconds. This is considerably faster than competing polarization sensors which acquire images sequentially (~10Hz) and then perform calculations.

2.3 Advantages of the Holographic Stokesmeter

In order to quantify the speed of the HSM, each component (optical element, CCD response, and signal addition) must be examined. Consider first the response of the holographic optical element, which is determined by the channel bandwidth. The holograms can be tailored for many bandwidths, but a typical value would be 1 nm, or 100 GHz. This corresponds to an optical response time of ~ 10 pico-seconds. Signal manipulation also does not significantly limit the response time of the HSM. Because of the nature of the calculations that are made, the processing can be accomplished with a programmable logic array (PLA) or a field programmable gate array (FPGA) since no dynamic decision-making is necessary. The relationship between the observed intensities and the Stokes components are set by the hologram. Such circuits perform calculations at
roughly the speed of the logic gates, where typical switching times are on the order of a nanosecond. The CCD arrays provide the true limit in response of the system. The speed of the CCD arrays is determined by the desired signal to noise ratio (SNR). The value of the signal and the noise are given by:

$$S \propto \eta I \tau; \quad N \propto \sqrt{\eta I \tau}; \quad \frac{S}{N} \propto \sqrt[4]{\eta I \tau}$$  \hspace{1cm} (2.1)

where $S$ is the signal, and $N$ is the noise, $\eta$ is the quantum efficiency of the CCD, and $\tau$ is the average collection time. The noise is also dependent on several factors including scattered intensity, CCD dark current, etc. A typical acceptable value for acceptable SNR is ~10. In this way the value of SNR determines the relationship between the available intensity and the response time. It is also reasonable to note that in our system, the limiting factor is an element common to every available technique – but in the other systems the optical elements set the maximum response time. Besides our speed advantage, our polarization separation element requires no power and contains no moving parts. These factors give our technique a significant competitive technical advantage. One possible consequence of dividing the initial image into four beams is that the intensity incident upon each CCD array will be lower than if the image was undivided. Whether this is a significant factor needs further investigation. Our design is also spectrally specific. That is, each implementation is a wavelength specific because the Stokes parameter determination is based upon Bragg reflection in thick holograms. This is an advantage since the system automatically filters input.
Chapter 3

Background

3.1 Derivation of Stokes Parameters and Mueller Matrices

Light of any polarization can be represented by its Stokes vector [5]. This vector has the advantage over the Jones vector of being able to represent partially polarized light as well as completely unpolarized light. The Mueller matrix of a medium characterizes the way the medium reflects light by providing a method of calculating the output Stokes vector given the matrix and the input Stokes vector [5,12].

3.1.1 Stokes Parameters (I,Q,U,V)

We define the Stokes vector in terms of the transverse components $E_x$ and $E_y$ of the electric field $\mathbf{E} = \mathbf{\hat{e}}_x E_x(t) + \mathbf{\hat{e}}_y E_y(t)$. In the general case, $E_x$ and $E_y$ have time-varying amplitudes and phases as follows:

$$E_x(t) = E_{x0}(t)e^{i\omega t + \phi_x(t)}$$

$$E_y(t) = E_{y0}(t)e^{i\omega t + \phi_y(t)}$$

The Stokes parameters can then be defined:

$$S = \begin{pmatrix}
I \\
Q \\
U \\
V
\end{pmatrix} = \begin{pmatrix}
\langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle \\
\langle E_x E_y^* \rangle - \langle E_y E_x^* \rangle \\
\langle E_x E_y^* \rangle + \langle E_y E_x^* \rangle \\
i\langle E_x E_y^* \rangle + i\langle E_y E_x^* \rangle
\end{pmatrix} = \begin{pmatrix}
\langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle \\
\langle E_x E_y^* \rangle - \langle E_y E_x^* \rangle \\
\langle E_x E_x^* \rangle - \langle E_y E_y^* \rangle \\
\langle 2E_x E_y \cos \phi \rangle
\end{pmatrix}$$

$$\langle E_x E_y \rangle = \begin{pmatrix}
2E_x E_y \sin \phi
\end{pmatrix}$$
where $\langle \cdot \rangle_T$ indicates the time average. For quasi-monochromatic light in the slowly-varying envelope approximation, $E_{x0}(t) \approx E_{x0}$ and $E_{y0}(t) \approx E_{y0}$. The parameters then become:

$$
I = E_x^2 + E_y^2 \\
Q = E_x^2 - E_y^2 \\
U = 2E_xE_y\langle \cos(\varphi(t)) \rangle_T \\
V = 2E_xE_y\langle \sin(\varphi(t)) \rangle_T \\
\varphi(t) = \phi_x(t) - \phi_y(t)
$$

\[(3.3)\]

$$
I^2 = Q^2 + U^2 + V^2 \quad (3b) \\
\varphi = \varphi_y - \varphi_x \quad (3c)
$$

where $I$ is the overall intensity (sum of the squares of the horizontal, $E_x$ and vertical, $E_y$, electric field amplitudes), $Q$ is the intensity difference between vertical and horizontal linear polarization, $U$ is the intensity difference between linear polarization at $+45^\circ$ and $-45^\circ$, and $V$ is the intensity difference between left and right circular polarization. The brackets in (3.2) indicate averaging with respect to time. It should be noted that when we define the Stokes vector for light we must define the $x$ and $y$ axis – which must be orthogonal to the direction of propagation since light is a transverse wave. The choice of the axis is arbitrary, but must be used consistently throughout the calculations. (3b) shows that the total intensity must equal the sum of the component intensities, and (3c) indicates the phase relationship between the $x$ ($\varphi_x$) and $y$ ($\varphi_y$) waves.

For completely unpolarized light, the phases $\phi_x(t)$ and $\phi_y(t)$ are fully random; that is, there is no correlation between them. Partially polarized light has a finite amount of correlation between the two phases, and fully polarized light has full correlation between $\phi_x(t)$ and $\phi_y(t)$. In other words, for fully polarized light, the difference between $\phi_x(t)$ and
\( \phi_j(t) \) is constant.

To illustrate this concept more clearly, we can make use of Fourier analysis. Consider the time-average function

\[
\frac{1}{T} \int_0^T f(t) dt
\]  

(3.4)

as a low-pass filter in the Fourier domain with width \( 1/T \). As \( T \) gets larger, the filter grows narrower, centered at zero-frequency. Now suppose the incident beam of light has components \( E_x \) and \( E_y \) with phases that vary randomly. The Fourier transform of this phase difference will have little to no zero-frequency (DC) component, and therefore the time-average (represented by the low-pass filter) will be approximately zero.

If instead the phases of \( E_x \) and \( E_y \) vary randomly but with some finite correlation between them, there will be a finite DC component in the Fourier domain. Thus the time-average will be non-zero, but not large due because the result of the integral is still divided by \( T \). Finally, consider the case of full correlation between the phases. In this case, the phase difference is constant, and the Fourier component is simply a delta function at zero-frequency. The time-average is now non-zero and not necessarily small. With these definitions in hand, we can turn to specific cases of the Stokes vector. For completely unpolarized light, there is no preference for \( E_x \) or \( E_y \), so \( Q = 0 \). As discussed above, with no correlation in the phases of \( E_x \) and \( E_y \) the time-average will be zero, and thus \( U \) and \( V \) are also zero. The Stokes vectors for horizontal, vertical, 45°, and circularly polarized light follow easily as shown in Figure 3.1.
3.1.2 Mueller Matrix: $M$

We can use the Stokes parameters in conjunction with the 4x4 element Mueller matrices to determine how the polarization of light changes through a certain medium. We derive the Mueller matrix below for light incident at a boundary between two materials of differing index of refractions. We proceed by considering individually four cases of different polarizations incident on the surface. Each case leads to four equations, and all sixteen equations allow us to solve for the 16 elements of the Mueller matrix. The relationship between input and output Stokes parameters is as follows:

$$
\begin{bmatrix}
  m_{11} & m_{12} & m_{13} & m_{14} \\
  m_{21} & m_{22} & m_{23} & m_{24} \\
  m_{31} & m_{32} & m_{33} & m_{34} \\
  m_{41} & m_{42} & m_{43} & m_{44}
\end{bmatrix}
\begin{bmatrix}
  I_i \\
  Q_i \\
  U_i \\
  V_i
\end{bmatrix}
= 
\begin{bmatrix}
  I_o \\
  Q_o \\
  U_o \\
  V_o
\end{bmatrix}
$$  

(3.5)

where the subscript $i$ is for initial and subscript $o$ is for output. We continue with the case of completely unpolarized light in order to determine the first row of the matrix. For fully unpolarized light:
The input Stokes vector has only $I \neq 0$. $t^2_x$ is the transmission coefficient squared for the x-component, and $t^2_y$ is the transmission coefficient squared for the y-component.

Because these two are not necessarily equal, the output Stokes vector has a $Q$ component.

Following this method, we write the matrix equations for horizontal (x-component only), 45°, and right-circularly polarized light as follows:

**Horizontal (x-component only)**

\[
\begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} \\
m_{21} & m_{22} & m_{23} & m_{24} \\
m_{31} & m_{32} & m_{33} & m_{34} \\
m_{41} & m_{42} & m_{43} & m_{44}
\end{bmatrix}
\begin{bmatrix}
E^2_x + E^2_y \\
0 \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
t^2_x E^2_x \\
t^2_x E^2_y \\
te^2_x E^2_x - t^2_y E^2_y \\
0 \\
0
\end{bmatrix}
\]

(3.6)

We have assumed that the light is fully polarized in these three cases, and $\varepsilon_x - \varepsilon_y = 0$ for 45° polarized light, and $\varepsilon_x - \varepsilon_y = \pi/2$ for right-circularly polarized light. It is evident that
the first matrix equation (3.7) will yield four equations of \(m_{11}, m_{12}, m_{13},\) and \(m_{14}\). Each of

the other three sets of matrices also generates four equations, and when all 16 equations

are solved, all 16 elements can be found:

\[
\begin{bmatrix}
    t_x^2 + t_y^2 & t_x^2 - t_y^2 & 0 & 0 \\
    t_x^2 - t_y^2 & t_x^2 + t_y^2 & 0 & 0 \\
    0 & 0 & 2t_xt_y & 0 \\
    0 & 0 & 0 & 2t_xt_y \\
\end{bmatrix}
\]  

(3.8)

The Mueller matrices for polarizers, wave-plates, and holograms can be determined in the

same manner.

### 3.2 Holography

The core component for the HSM is a holographic diffraction element (HDE). In this

section we will provide some background into holographic theory and our material

selection.

*Figure 3.2 A brief description of a thick hologram*
Conceptually, a hologram can be thought of as an object that, when one of the two original beams is introduced, the continuation of the complimentary beam is the result. This idea is presented in Figure 3.2; in the far left panel the hologram is written using a “reference” and an “object” beam. In the center panel the object beam is used to illuminate or “read” the hologram and a beam exits the sample in the same direction as the original reference beam. On the right the reference beam is used to read the hologram, re-creating the object beam. Physically, the incident ray is diffracted by gratings created inside of the holographic material (for volume holograms this is the result of two wave mixing). The gratings are formed by an intensity interference pattern inside the hologram which turns into a modulation of the index of refraction of the material. The thickness of the substrate and the depth of the index modulation determine the maximum number of holograms, or gratings that can be written in one volume. Conventional holographic materials are limited in the depth to a few hundred microns. The depth of the index modulation is typically characterized by a material parameter; $M_\#$, which corresponds to the maximum number of 100% efficiency holograms that can be written in a single location [23]. The efficiency of each of $N$ weak holograms written in a single location is then given by $(M_\#/N)$. Typical materials have an $M_\#$ of the order of unity. For this thesis, we use a thick material; Memplex\textsuperscript{TM}, developed by Laser Photonics Technology, Inc., of Amherst, NY [24]. An image of the material is displayed in Figure 3.3. Memplex\textsuperscript{TM} is a polymer substrate that is doped with a dye that is sensitive to a particular wavelength of light. When the dye is exposed to the laser it converts the light intensity variation into a polymerization variation inside the material gratings. Since our material is a polymer, it can be formed into any desired thickness, with substrates typically on the order of
millimeters (as opposed to microns). Memplex™ also has an \( M_\parallel \) an order of magnitude greater than typical holographic materials. This allows us to write multiple high-efficiency holograms in the same volume.

3.2.1 Holographic Grating

A simple sinusoidal holographic grating can be written by two monochromatic plane waves \( E_1 = \hat{e}_1 A_1 e^{i(\omega t - k_1 \cdot r)} \) and \( E_2 = \hat{e}_2 A_2 e^{i(\omega t - k_2 \cdot r)} \), interfering inside the photosensitive material. For simplicity, assume that \( A_1 \) and \( A_2 \) are real, and the waves have the same polarizations \( \hat{e}_1 \cdot \hat{e}_2 = 1 \). The intensities of the two beams are \( I_1 = |A_1|^2 \) and \( I_2 = |A_2|^2 \). Intensity distribution of the interference pattern inside the material is given as

\[
I \propto |E_1 + E_2|^2 = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(K_G \cdot r) = I_0[1 + m \cos(K_G \cdot r)]
\]

where

\[
K = k_1 - k_2
\]

(3.10) is the grating vector, \( I_0 = I_1 + I_2 \), and

\[
\text{Figure 3.3 Image of Memplex™ holographic material.}
\]
\[ m = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} \]  

(3.11) is the grating modulation depth. The spatial pattern of the intensity profile creates a grating in the holographic material by modifying the index of refraction of the material, so that \( n = n_0 + \Delta n \), where \( n_0 \) is the refractive index prior to illumination. The holographic grating is defined to be the spatial modulation of the refractive index of the material

\[ \Delta n = n_1 \cos(K \cdot r) \]  

(3.12)

where \( n_1 \) is the amplitude of the modulation, and the periodicity of the index modulation given as

\[ \Lambda = \frac{2\pi}{K} \]  

(3.13)

(3.13) is known as the grating wavelength. This grating can diffract light. The writing process of the holographic grating is shown in Figure 3.4(a). In general, angles of incidence are not equal (\( \theta_1 \neq \theta_2 \)), so that the grating is slanted at an angle \( \phi \). When the grating is illuminated with the plane wave in the direction of \( k_1 (k_2) \), the plane wave in

\textit{Figure 3.4 Holographic grating (a) writing and (b) readout}
the direction of \( \mathbf{k}_z (\mathbf{k}_i) \) is reconstructed. Figure 3.4(b) shows reconstruction of the plane wave in the direction of \( \mathbf{k}_z \).

The holographic grating can be expressed as the modulation of the dielectric constant of the material

\[
\varepsilon = \varepsilon_0 + \varepsilon_1 \cos(\mathbf{K} \cdot \mathbf{r}) \quad (3.14)
\]

Since Mempex\textsuperscript{TM} material we use has typical values of \( n_i \sim 10^{-3} \) to \( 10^{-4} \) and \( n_0 \approx 1.5 \), the weak modulation condition \( n_i \ll n_0 \) is usually valid. This allows us to relate the modulation of the refractive index to the modulation of the dielectric constant of the material as following

\[
n = \sqrt{\varepsilon} = \left( \varepsilon_0 + \varepsilon_1 \cos(\mathbf{K} \cdot \mathbf{r}) \right)^{1/2} = \\
= \sqrt{\varepsilon_0} + \frac{\varepsilon_1}{2\sqrt{\varepsilon_0}} \cos(\mathbf{K} \cdot \mathbf{r}) = n_0 + n_1 \cos(\mathbf{K} \cdot \mathbf{r}) \quad (3.15)
\]

where the Taylor series expansion in the first order was made for this weak modulation condition. Thus the average refractive index is \( n_0 = \sqrt{\varepsilon_0} \), and the amplitude of the spatial modulation of the refractive index is \( n_1 = \frac{\varepsilon_1}{2\sqrt{\varepsilon_0}} \).

3.2.2 Diffraction from Periodic Media

Now, we consider the scattering process of a monochromatic plane wave from a periodic medium. Consider an extreme situation when the index modulation is lumped into an array of equidistant planes as depicted in Figure 3.5. In addition, we assume that these planes are infinite so that reflections from these planes are mirror-like with angle of
reflection equal to angle of incidence. Each of the planes reflects only a very small fraction of the incident plane wave. The scattered light consists of linear

![Figure 3.5: Scattering of monochromatic plane wave from a periodic medium](image)

superposition of all these reflected plane waves. The diffracted beams are found when all these reflected plane waves add up constructively.

Let $\Lambda$ be the spacing between these planes. This is also the period of the index variation. The path difference for rays reflected from two adjacent planes is $2\Lambda \sin \theta$, where $\theta$ is the angle between the ray and the planes. Constructive interference occurs when the path difference is an integral number of wavelengths $\lambda/n$ in the medium: \[ 2\Lambda \sin \theta = N(\lambda/n) \] (3.16)

where $n=n_0$ is the spatially averaged index of refraction of the medium and $N$ is an integer. Eq.(3.16) is known as the Bragg condition. The beam diffraction occurs only for certain values of $\theta_n$ which obey the Bragg condition so that reflections from all planes add up in phase. Equation (3.16) can also be written as
\[ 2k \sin \theta_B = NK \] (3.17)

where \( k \) is the wave number of the light beam in the medium \( (k = 2\pi n/\lambda) \) and \( K = 2\pi/\Lambda \) is the grating wave number. The left side of Equation (3.17) is the change of the wave vector upon diffraction from the periodic medium. This periodic grating modifies the energy–momentum relation of light wave due to Bragg diffraction. Then, the change of wave vector is exactly an integral number of the grating wave vector. We can also write the Bragg condition as

\[ \sin \theta_B = \frac{\lambda}{2n\Lambda} \] (3.18)

Note that there are many Bragg angles, each corresponding to a different value of \( N \).

### 3.3 Thin vs Thick Grating

The terms thin and thick gratings may be ambiguous. However, meaningful definitions are possible based on either the diffraction regime [25] or on the angular and wavelength selectivity [26].

**Bragg and Raman-Nath Diffraction Regimes**

If we decompose the periodic modulation of the index of refraction

\[ n = n_0 + n_1 \cos(K \cdot r) \] (3.19)

into its Fourier components, we obtain

\[ n(r) = n_0 + n_1 \sum a_m e^{imKr} \] (3.20)
where \( a_m \) is the \( m \)th Fourier component of the periodic index variation. We note that the \( m \)th Fourier component has a wave number of \( mK \). Each of the Fourier components contributes to a diffraction order. In Bragg regime only the first order diffraction is allowed, whereas in Raman-Nath diffraction regime multiple diffraction orders can exist. For the case shown in Figure 3.6, Bragg diffraction is described by Equation (3.17) with \( N = \pm 1 \) or \( -1 \). It can be shown by rigorous analysis that the distinction between Bragg and Raman-Nath regimes depends on grating thickness and the magnitude of the refractive index modulation.

Consider the case of weak refractive index modulation (e.g. \( n_i < 10^{-3} \)). Since the holographic material has a finite width, there is an uncertainty about the direction of the wave vector \( K \) of the recorded grating. However, conservation of energy dictates that the magnitude of \( K \) is finite. Multiple scattering will occur if the angular distribution of the grating wave vector \( K \) is large compared to the Bragg angle \( \theta_n \). Let \( L \) be the width of the grating. The angle of spreading of the grating wavevector \( \psi \) is determined from the uncertainty bandwidth product as \( \Delta KL = 2\pi \), that is,

\[
\sin(\psi/2) = \Delta K / 2K
\]

(3.21)

![Figure 3.6 Angular spread of the grating wave vector](image)

Assuming that \( \Delta K \ll K \), we have \( \psi \approx \Delta K / K \), then
\[ \psi \approx \frac{\Lambda}{L} \]  

(3.22)

The Bragg angle is given approximately by (3.18)

\[ \theta_B \approx \frac{\lambda}{2n_0\Lambda} \]  

(3.23)

We now define a dimensionless parameter \( Q \) (also known as Klein and Cook parameter)

\[ Q \equiv \frac{4\pi\theta_B}{\psi} \]  

(3.24)

then,

\[ Q = \frac{2\pi\lambda L}{n_0\Lambda^2} \]  

(3.25)

It is customary to defines the regime of \( Q > 10 \) as the Bragg regime of optical diffraction.

Note that \( Q \propto \frac{L}{\Lambda} \), so that a large \( Q \) means that the transverse length of the sample is much longer then the grating wavelength. In a thin grating, the transverse dimension of the periodic index variation is relatively small compared with the beam size and/or wavelength of light. When a plane wave is incident into the periodic medium, the diffraction from each of these planes is a result of the finite size of the planes.

However, a large \( Q \) does not guarantee Bragg diffraction. It has been shown that for large refractive index modulation, gratings with \( Q > 10 \) produce multiple diffraction orders. The amplitude of the refractive index modulation \( n_1 \) can be introduced through the parameter called the grating strength. In case of equal angle of incidence \( \theta \) for both writing beams, the grating strength \( \nu \) is

\[ \nu = \frac{\pi n_1 L}{\lambda \cos \theta} \]  

(3.26)

We define the dimensionless parameter \( \rho \) (also known as Raman-Nath parameter)
\[ \rho \equiv Q/(2\nu \cos \theta) \]  \hspace{1cm} (3.27)

then,

\[ \rho = \frac{\kappa^2}{\Lambda^2 n_0 n_i} \]  \hspace{1cm} (3.28)

It is customary to define the regime where \( \rho \gg 1 \) as the Bragg regime. Note that \( \rho \) is independent of the grating width \( L \). Since \( \rho \propto \frac{1}{n_i} \), it is clear that for weak modulation \( \rho \) is large, and conversely. Therefore, if \emph{thin} grating is intended to mean Raman-Nath regime diffraction, then the required condition is \( Q' \gamma \leq 1 \). If \emph{thick} grating is intended to mean Bragg regime diffraction, then the required condition is \( \rho > 10 \). (If \( \eta_1 > \eta_{-1} \) is also included in the definition of Bragg regime, \( Q' > 1 \) is required in addition.)

\textit{The angular and wavelength selectivity}

A \emph{thin} grating may be described as a grating exhibiting relatively little angular and wavelength selectivity. As the incident wave is dephased (either in angle incidence or in wavelength) from the Bragg condition, the diffraction efficiency decreases. The angular range or wavelength range for which the diffraction efficiency decreases to half of its on-Bragg-angle value is determined by the thickness of the grating \( d \) expressed as a number of grating period \( \Lambda \). For a \emph{thin} grating this number may be reasonably chosen to be

\[ d / \Lambda < 10 \]  \hspace{1cm} (3.29)

The region of \emph{thin} grating behavior according to the angular- and wavelength-selectivity-based definition (3.9) is depicted in Figure 3.7. Gratings having angular and wavelength
selectivities with FWHM wider than that for \( d/\Lambda = 10 \) may be considered to be thin gratings.

A thick grating may conversely be described as a grating exhibiting strong angular and wavelength selectivity. A relatively small change in the angle of incidence from the Bragg angle or a relatively small change in the wavelength at the Bragg angle produces significant dephasing and the diffraction efficiency decreases correspondingly. Thick grating behavior may be considered to occur when

\[
d / \Lambda > 10 \tag{3.30}
\]

This is the angular-and-wavelength-selectivity-based definition of a thick grating. The region of this behavior is also shown in Figure 3.7.

**Figure 3.7** Typical angular selectivity plots (normalized first-order diffraction efficiency vs angular deviation from the Bragg angle) for various values of \( d/\Lambda \)
That is, if *thin* grating is intended to mean broad angular and wavelength selectivity, then the required condition is $d/\Lambda < 10$. If *thick* grating is intended to mean narrow angular and wavelength selectivity, then the required condition is $d/\Lambda > 10$. 
Chapter 4

Theoretical Analysis

The HSM takes advantage of the polarization sensitivity of the diffraction efficiency of holographic gratings. In this section we will provide a step-by-step description of how this calculation occurs.

4.1 Polarization Analysis – Transmission Across Front Surface

\[ M_T = M_{ES} * M_K * M_{FS} \]

*Figure 4.1* A series of Mueller matrices that describe the transformation of the input Stokes vector \( \mathbf{S}_i \).

*Figure 4.2* Illustration of Fresnel transmission and reflection for in plane (a.) and out of plane (b.) polarized light.
Figure 4.1 illustrates schematically the progression of the light through our system. We can represent this same process mathematically, by a series of Mueller matrices. The incident polarization state is scattered by each matrix, and the final diffracted intensities can then be used to determine the initial Stokes parameters. This process is shown in Figure 4.2.

Firstly, we must take into account interaction between the initial polarization state $S_0$ and the front surface. This interaction is governed by the Fresnel reflections – which ensure that the electric field component that is tangential to the interface is conserved. The result is that linearly polarized (LP) light lying in the plane of incidence (Figure 4.2(a.)) is treated differently than LP light perpendicular to the plane of incidence (Figure 4.2(b.)). The plane of incidence is defined as the plane containing the propagation vectors of the incident light, reflected light and transmitted light. The reflection and transmission coefficients for each component amplitude are given by:

$$r_\parallel = \frac{E_{0R}}{E_{0I}} = \frac{n_r \cos \theta_i - n_i \cos \theta_r}{n_i \cos \theta_r + n_r \cos \theta_i} \quad (4a)$$
$$t_\parallel = \frac{E_{0T}}{E_{0I}} = \frac{2n_i \cos \theta_i}{n_i \cos \theta_r + n_r \cos \theta_i} \quad (4b)$$
$$r_\perp = \frac{E_{0R}}{E_{0I}} = \frac{n_i \cos \theta_i - n_r \cos \theta_r}{n_i \cos \theta_r + n_r \cos \theta_i} \quad (4c)$$
$$t_\perp = \frac{E_{0T}}{E_{0I}} = \frac{2n_i \cos \theta_i}{n_i \cos \theta_r + n_r \cos \theta_i} \quad (4d)$$

where $n_i$, $n_r$, are the indices of reflection outside and inside the media, respectively, and $\theta_i$, and $\theta_r$ represent the incident and transmission angles, all of which satisfy Snell’s law:

$$n_i \sin \theta_i = n_r \sin \theta_r \quad (4.1)$$

We can then calculate the Stokes vector that is transmitted ($S_T$) through the front surface:
\[
\begin{pmatrix}
  t_{\|}^2 + t_{\perp}^2 & t_{\|}^2 - t_{\perp}^2 & 0 & 0 \\
  t_{\|}^2 - t_{\perp}^2 & t_{\|}^2 + t_{\perp}^2 & 0 & 0 \\
  0 & 0 & 2t_{\|}t_{\perp} & 0 \\
  0 & 0 & 0 & 2t_{\|}t_{\perp}
\end{pmatrix}
\begin{pmatrix}
  I_0 \\
  Q_0 \\
  U_0 \\
  V_0
\end{pmatrix}
= \begin{pmatrix}
  I_T \\
  Q_T \\
  U_T \\
  V_T
\end{pmatrix}
= \begin{pmatrix}
  I_0(t_{\|}^2 + t_{\perp}^2) + Q_0(t_{\|}^2 - t_{\perp}^2) \\
  I_0(t_{\|}^2 - t_{\perp}^2) + Q_0(t_{\|}^2 + t_{\perp}^2) \\
  2t_{\|}t_{\perp}U_0 \\
  2t_{\|}t_{\perp}V_0
\end{pmatrix}
\]

(4.2)

where the matrix elements are defined in equation (4). \( S_T \) is the vector that interacts with the holographic gratings.

### 4.2 Polarization Interaction with Holographic Gratings

In order to determine the interaction between the holographic grating, and the incident and diffracted light we must solve the coupled wave equations. We will see that light interacts, or reflects off of holographic gratings in a way analogous to the interaction with a surface. In the following section we will present the solution to the wave equation (which follows the analysis by Kogelnik) [22], and the resulting matrix, which will indicate why we require multiple gratings for complete Stokes parameter calculation. For the sake of simplicity, we will assume zero absorption and that we are always working at the Bragg condition (note: the addition of these corrections would take place automatically during an experimental evaluation of the hologram). We will then present the solution to the wave equation for the case of two gratings. Figure 4.3 illustrates the geometry of our model, where the incident wave \( R \) has a

![Figure 4.3](image-url)
an angle \( \theta_r \), and the grating vector \( \mathbf{K} \) makes an angle \( \phi \) with respect to the \( z \) axis. We start the analysis by recognizing that the electromagnetic wave in the media must satisfy the wave equation:

\[
\nabla^2 \mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) + k^2 \mathbf{E} = 0
\]

\[
\mathbf{E}(\mathbf{r}) = \mathbf{R}(z)e^{-i\beta z} + \mathbf{S}(z)e^{-i\kappa z}, \quad k^2 = \beta^2 + 2\beta\kappa(e^{-i\mathbf{K} \cdot \mathbf{z}} + e^{i\mathbf{K} \cdot \mathbf{z}})
\]

\[
\beta = 2\pi n / \lambda \quad \kappa = \pi n_1 / \lambda
\]

where we have represented the wave \( \mathbf{E} \) as a superposition of the incident (\( \mathbf{R} \)) and the diffracted (\( \mathbf{S} \)) waves with complex amplitudes. Note that we have not made any assumptions about the polarization of \( \mathbf{R} \) and \( \mathbf{S} \) since we have represented the amplitudes as vectors. The exponential terms of the propagation constant \( k \) contain the spatial modulation of the diffraction grating (\( \mathbf{K} \)) and an average propagation constant \( \beta \) which is equal to the wave vector of the incident light (inside the material), and the coupling constant \( \kappa \) which couples the incident light into the diffracted beam where we are now representing the average index of refraction with \( n \), and the spatial modulation of the index of refraction by \( n_1 \). As stated earlier, the efficiency of the grating is related to \( n_1 \). The Bragg condition which leads to strong diffraction can be expressed by the vector relationship:

\[
\mathbf{\sigma} = \mathbf{\rho} - \mathbf{K}
\]

We can simplify the notation if we assume that the polarization does not change in the grating and rewrite \( \mathbf{R} \) and \( \mathbf{S} \) as:

\[
\mathbf{R}(z) = R(z)\hat{\mathbf{r}} \quad \mathbf{S}(z) = S(z)\hat{\mathbf{s}}
\]
where \( R \) and \( S \) are now scalar amplitude functions multiplied by unit polarization vectors. We have also assumed that \( R \) and \( S \) are transverse waves, meaning the polarization vectors are perpendicular to the direction of propagation, or:

\[
\begin{align*}
\mathbf{p} \cdot \mathbf{R} &= (\mathbf{p} \cdot \mathbf{\hat{r}}) = 0 \\
\mathbf{\sigma} \cdot \mathbf{S} &= (\mathbf{\sigma} \cdot \mathbf{\hat{s}}) = 0 \\
(\mathbf{\hat{r}} \cdot \mathbf{\hat{s}}) &= 1
\end{align*}
\]  

By inserting (4.4) into (4.5), enforcing the Bragg condition (4.6) in the exponentials, and equating terms with equal phase, we arrive a pair of coupled equations:

\[
\begin{align*}
R^* - 2iR' \rho_z + 2\kappa \beta S(\mathbf{\hat{r}} \cdot \mathbf{\hat{s}}) &= 0 \\
S^* - 2iS' \sigma_z + 2\kappa \beta R(\mathbf{\hat{r}} \cdot \mathbf{\hat{s}}) &= 0
\end{align*}
\]

where the primes represent differentiation with respect to \( z \). If we apply the slowly varying envelope approximation (SVEA) which states the both \( R \) and \( S \) vary slowly with respect to \( z \) we can set \( R'' \) and \( S'' \) equal to zero and rewrite (4.9) as:

\[
\begin{align*}
c_R R' &= -i\kappa (\mathbf{\hat{r}} \cdot \mathbf{\hat{s}}) S \\
c_S S' &= -i\kappa (\mathbf{\hat{r}} \cdot \mathbf{\hat{s}}) R
\end{align*}
\]

\[
\begin{align*}
c_R &= \rho_z / \beta = \cos \theta_t \\
c_S &= \sigma_z / \beta = \cos \theta_t - \frac{K}{\beta} \cos \phi
\end{align*}
\]

where we have introduced \( c_R \) and \( c_S \), which are known as obliquity factors, the presence of which indicate that energy flow is conserved in the \( z \) direction. By solving the differential equations and matching the boundary conditions:

\[
R(z = 0) = R_0 \quad S(z = 0) = 0
\]

we arrive at the solution for the diffracted intensity as a function of \( z \):

\[
S(z) = -i \frac{\kappa (\mathbf{\hat{r}} \cdot \mathbf{\hat{s}})}{c_S \alpha_0} R_0 \sin(\alpha_0 z)
\]

\[
\alpha_0 = \sqrt{\frac{(\kappa (\mathbf{\hat{r}} \cdot \mathbf{\hat{s}}))^2}{c_R c_S}}
\]
As we stated in the description of polarized light, any polarization of light can be decomposed into two, orthogonally related components. Therefore, just as we did at the entrance surface of the material, we can view the light incident upon the hologram $R$, as having perpendicular and planar components. In the perpendicular case the incident and diffracted rays have the same polarization direction, so the dot product is equal to unity from (4.8). For in-plane polarization, we use the definition of the relationship between the incident ray and the diffracted ray directions. This provides us with two diffraction amplitudes:

$$S(d)_\perp = -i \frac{c_L}{c_S} R_0 \sin \left( \frac{\kappa^2}{c_R c_S} d \right)$$

$$S(d)_\parallel = -i \frac{c_L}{c_S} R_0 \sin \left( \frac{-\kappa^2 \cos 2(\theta_T - \phi)}{c_R c_S} d \right)$$

These relationships are depicted schematically in Figure 4.4 for out of plane (a) and in-plane (b) polarizations. If we assume unitary input ($R^2 = 1$) then we can write this polarization dependence into a matrix form, which is very similar to (4.2):

$$S_D = M_S S_T = \frac{1}{2} \begin{pmatrix} S_\perp^2 + S_\parallel^2 & S_\parallel^2 - S_\perp^2 & 0 & 0 \\ S_\perp^2 - S_\parallel^2 & S_\parallel^2 + S_\perp^2 & 0 & 0 \\ 0 & 0 & 2S_\parallel S_\perp & 0 \\ 0 & 0 & 0 & 2S_\parallel S_\perp \end{pmatrix} \begin{pmatrix} I_T \\ Q_T \\ U_T \\ V_T \end{pmatrix} = \begin{pmatrix} I_D \\ Q_D \\ U_D \\ V_D \end{pmatrix}$$

Our goal is to identify the four original Stokes parameters. We propose to do this using only different values of $I_D$. Using two different matrices of the above form will allow us to resolve $I_T$ and $Q_T$. In order to determine the other values, we must have a matrix with non-zero off-diagonal elements. We can do this by using a holographic grating that is
rotated about the $z$ axis. Then, in order to use the same procedure used above, we must rotate the polarization axis of the incident light. In this way we can again describe the incident polarization in terms of in-plane and out-of-plane components. This is described schematically in Figure 4.5. Rotation of the polarization axis is achieved by a rotation matrix:

$$M_R(\gamma) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos 2\gamma & -\sin 2\gamma & 0 \\
0 & \sin 2\gamma & \cos 2\gamma & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \quad (4.18)$$

This rotation matrix mixes the $Q$ and $U$ components, which will allow them to be determined using the grating matrices like (4.17). We either transform the incident vector with (4.18) or bring the rotation into the grating matrix (4.17) through matrix multiplication.

In order to determine the interaction between multiple holographic gratings with the incident wave, we must revisit the wave equation. This time we will solve for two
diffracted waves. We choose this number because four gratings will give us four equations, which is the minimum we need in order to solve for the four unknown Stokes parameters. Again, we start from (4.3), this time we assume the electric field inside the material is equal to the superposition of 5 waves:

\[ E(r) = R(z)e^{-i\beta z} + S_1(z)e^{-i\theta_1} + S_2(z)e^{-i\theta_2} \]  

(4.19)

where each \( S_n \) corresponds to a different grating vector \( K_n \). This appears in the altered form of (4.4):

\[ k^2 = \beta^2 + 2\beta \kappa_1 (e^{-iK_1z} + e^{+iK_1z}) + 2\beta \kappa_2 (e^{-iK_2z} + e^{+iK_2z}) \]  

(4.20)

Inserting (4.19) and (4.20) into the wave equation (4.3), we end up with relationships similar to (4.10) and (4.11):

\[ c_{R}' = -i\kappa_1 (\hat{r} \cdot \hat{s}_1) S_1 - i\kappa_2 (\hat{r} \cdot \hat{s}_2) S_2 \quad c_{S_j}' = -i\kappa_j (\hat{r} \cdot \hat{s}_j) R \]  

(4.21)

\[ c_R = \rho \beta = \cos \theta \quad c_{S_j} = \sigma \beta = \cos \theta_j - \frac{K_j}{\beta} \cos \phi_j \]  

(4.22)

These equations lead us to the same wave form as before, but with a different exponential constant that interrelates the diffracted amplitudes:

\[ \alpha_0 = \sqrt{\frac{1}{c_R} \left[ \frac{(\kappa_1 (\hat{r} \cdot \hat{s}_1))^2}{c_{S1}} + \frac{(\kappa_2 (\hat{r} \cdot \hat{s}_2))^2}{c_{S2}} \right]} \]  

(4.23)

And then for the diffracted amplitude at the exit surface we have:

\[ S_j(d) = -i \frac{\kappa_j (r \cdot s_j)}{c_{Sj} \alpha_0} R_0 \sin (\alpha_0 d) \]  

(4.24)

Using these equations, we can build a matrix for each diffracted beam. Since each matrix is rotated about the z axis, \( I_f \) will be related to \( I_T, Q_T, \) and \( U_T. \)
\[ I_1 = a_1 I_r + b_1 Q_r + c_1 U_r \]
\[ I_2 = a_2 I_r + b_2 Q_r + c_2 U_r \]  \hspace{1cm} (4.25)

In order to solve the parameters completely, we require at least two more equations, and they must contain \( V_T \). This is achieved by examining the role of the quarter-wave plate. The effect of the QWP is to transform the identities of \( Q_T \), \( U_T \), and \( V_T \) because we are adding or subtracting \( \pi/2 \) to \( \epsilon \). If the slow axis of the QWP is aligned with the \( y \) axis, then:

\[
\epsilon_y \rightarrow \epsilon_y - \pi/2; \quad \epsilon \rightarrow \epsilon - \pi/2; \quad \cos(\epsilon) \rightarrow \cos(\epsilon - \pi/2) = \sin(\epsilon) \\
\sin(\epsilon) \rightarrow \sin(\epsilon - \pi/2) = -\cos(\epsilon)
\]  \hspace{1cm} (4.26)

which, from our definition of the Stokes parameters (2a) leads to \( V \rightarrow U, \ U \rightarrow -V \). The function of the QWP is depicted in Figure 4.6. After the incident beam has been shifted with the QWP, it is routed through two more rotated gratings, which produces the last two desired equations:

\[ I_3 = a_3 I_r + b_3 Q_r + c_3 V_r \]
\[ I_4 = a_4 I_r + b_4 Q_r + c_4 V_r \]  \hspace{1cm} (4.27)

---

**Figure 4.6** Illustration of quarter-wave plate (QWP) function. When the fast (F) axis of the QWP is aligned with the y axis (a.) the y-wave (blue) races ahead 1/4 cycle, which converts the right-circularly polarized light (RCP) into +45º linearly polarized light (LP). When F is aligned with the x axis (b.) the RCP is converted into -45º LP.
4.3 Analysis Summary

In the previous sections we have developed the mathematical tools necessary to determine the four Stokes parameters of light incident upon the system described in Figure 2.1. The incident light is “operated” upon by several matrices sequentially. This idea is illustrated by Figure 4.1. We must remember that the order of operations on a vector is from right to left. From basic linear algebra we know that we can consolidate all of the individual matrices into a single “Total” interaction matrix which is unique for each diffracted beam:

\[
M_{\text{TOT}} = M_{ES} M_{Kj} M_{a/z} \left( M_{QWP} \right) M_{FS}
\]  

(4.28)

where the parentheses around \( M_{QWP} \) indicate that this matrix is only included for two of the diffracted beams. The intensity of each beam will be related to several of the Stokes parameters by specific coefficients. These intensities form a linear system of equations that allow us to find the original Stokes parameters of the incident light. Alternatively, the correct superposition of the CCD images allows us to image only a pre-determined Stokes vector or polarization “signature”.
Chapter 5

Demonstration of a Volume Holographic Stokesmeter

5.1 Highly Polarization-Sensitive Thick Gratings

We have performed the polarization dependence of the diffraction efficiency of a single grating. The experimental setup is illustrated Figure 5.1. We used a 532 nm. frequency-doubled YAG laser as a source. The beam, (which is already polarized upon emission from the laser) was redirected by a polarizing beam-splitting cube. This beam was then passed through a half-wave plate (HWP) which was mounted on a rotational mount. In this way we are able to vary the orientation of the linearly polarized light used to probe the grating. Note that a rotation of $\theta$ by the HWP results in a $2\theta$ rotation of the incident linear polarization. The diffraction efficiency was monitored with the use of two photodetectors. The direct beam (unscattered) intensity $I_0$ was recorded as was the diffracted intensity $I_D$. The diffraction efficiency $\eta$ was calculated as:

![Figure 5.1 Schematic of experimental set-up to measure polarization sensitivity of diffraction efficiency](image)
\[
\eta = \frac{I_D}{I_D + I_0}
\]  

(5.1)

This measure eliminates, to first order, the effect of the Fresnel reflection losses of the incident beam, which were present because the sample was not anti-reflection coated. As we saw in the previous section, the Fresnel reflection coefficient is sensitive to polarization orientation, it is very important to eliminate this effect in order to observe the true polarization sensitivity of the grating. Also, in order to avoid a strong Fresnel effect the entrance and exit angles of the grating were tailored to avoid the Brewster angle.

The diffraction efficiency for general polarization follows from equation (4.15) and (4.16):

\[
\eta = \sin^2 \left( \kappa \left( \hat{u}_i \cdot \hat{u}_d \right) \frac{d}{\sqrt{\cos \theta_i \cos \theta_d}} \right)
\]

(5.2)

For the case of parallel polarization the dot product \( \hat{u}_i \cdot \hat{u}_d = -\cos(2(\theta_i - \phi)) \), and the equations for the diffraction efficiency of each polarization are as follows:

\[
\eta_{\perp} = \sin^2 \left( \frac{\pi n'}{\lambda} \frac{d}{\sqrt{\cos \theta_i \cos \theta_d}} \cos(2(\theta_i - \phi)) \right)
\]

\[
\eta_{\parallel} = \sin^2 \left( \frac{\pi n'}{\lambda} \frac{d}{\sqrt{\cos \theta_i \cos \theta_d}} \cos(2(\theta_i - \phi)) \right)
\]

(5.3)

Studies of the polarization dependence of the diffraction efficiency have been carried out for the case of achieving elimination of one unwanted polarization, using the holographic Brewster angle method, for the purpose of creating holographic optical elements [27, 28]. We are interested in establishing the precision with which the observed polarization diffraction contrast matches the analytic theory. In particular, this requires an indirect determination of the index modulation amplitude from the diffraction efficiency.
of one polarization. This value is then used to predict the diffraction efficiency at the other polarization, in order to compare with the experimental value.

The Memplex material we use is a dye-doped photopolymer with an index of refraction of 1.482 and a sample thickness of 2mm. Given the index of this material, it is not possible to achieve the required $\theta_i - \phi = 45^\circ$ condition using beams incident on the same surface. However, if one were to use a cubic geometry, the condition is easily achievable.

The gratings used here were written at 532nm with writing angles of 52.5 and 57 degrees. Reading was done at the same wavelength and at 57 degrees. Six individual gratings were written using different exposure times [29]. The diffraction efficiency was then measured for various angles of the polarization of the incident read beam. The value of $n'$ was calculated using the equation (5.2) with measured values of the diffraction efficiency and compensating for Fresnel reflection loss. The coupled-wave analysis is done without incorporating Fresnel reflection, and the Fresnel coefficients are derived for a medium without any gratings. We note from the agreement of our results that these quantities can be considered independently. The gratings were read at the Bragg angle, so the Fresnel angle of the incoming beam on the front surface is 57 degrees and the Fresnel angle of the diffracted beam on the exit surface is 52.5 degrees. Knowing this reflection loss and using equation (5.2), it is possible to break up an incoming beam of arbitrary polarization into its perpendicular and parallel components and calculate the expected diffraction. Taking care to incorporate that the half-wave plate used to collect the data rotates the field, not the intensity directly, the theoretical diffraction efficiency can then be mapped for a desired range of half-wave angles. The resulting theoretical and experimental results are plotted in figure 5.2 without using any free fitting parameter [29]. Note that the
experimental deviation from the coupled-wave theory is very small. These results also demonstrate a high degree of contrast in diffraction efficiency between the s- and p-polarizations.

Figure 5.2 Diffraction efficiency vs. Polarization. 0 and 90 degrees are s-polarized and 45 degrees is p-polarized. Dashed line indicates theoretical calculations.
5.2 Noise Analysis and Heterodyned Stokesmeter

We performed two separate numerical simulations in order to simulate the performance of the holographic Stokesmeter in the presence of noise [29]. The first case considered the effect of additive white Gaussian noise (AWGN) added to the four intensity measurements. The measurement matrix is assumed to be known without error. The second case considers no noise in the intensity measurements, but includes noise in the measurement matrix itself. For each of these cases a variety of Stokes vectors were tested. Shown here are the results for the Stokes vector \([1 \ -0.6 \ 0 \ 0.8]\) averaged over 200 runs. Percent error is plotted versus the contrast ratio of the gratings. The grating parameters used in the simulation were in favor of stronger diffraction of perpendicular polarization for the first and fourth grating and in favor of stronger diffraction of parallel polarization for the second and third grating. The rotation angles used were 5 and 40 degrees. Using these grating parameters leads to an improvement of the measurement matrix, however, these parameters do not necessarily represent the optimal set. A more exhaustive search through the entire parameter space is required to fully optimize the measurement matrix. The variance of the noise used was -25, -30, -35, and -40dB compared to a maximum normalized intensity of 1.

The results for case one are shown in Figure 5.3. We see from the figure that as the contrast ratio increases, the average percent error decreases, with limiting gains in the improvement past 50% contrast. These error results are specific to the chosen input Stokes vector, but the general trend is the same for an arbitrary input.

The results for case two are shown in Figure 5.4. This case shows the same trend of decreasing error as the contrast ratio increases. Note the unusually large error for
contrast ratio values less than 50% in this case. This is due to the fact that the noise is added to the measurement matrix in this scenario, and for average noise values that are larger than the difference between the parallel and perpendicular polarization components of the diffraction efficiency, the sign of the terms $A_i + B_i$ in equation 1 will change. This can lead to very large errors in the calculation. As the contrast ratio increases, this effect lessens and the percent error rates approach normal values. As we can see from the data in Figures 5.3 and 5.4, for contrast ratios of greater than 50%, a relatively noise-tolerant

Figure 5.3 Percent error plotted versus contrast ratio for each Stokes parameter for the case of AWGN in the intensity measurements. The separate lines in each graph represent the different noise levels: $+ = -25\text{dB}$, $\circ = -30\text{dB}$, $\times = -35\text{dB}$, $\Box = -40\text{dB}$, $\ast = \text{no noise}$.

Figure 5.4 Percent error plotted versus contrast ratio for each Stokes parameter for the case of AWGN in the measurement matrix. The separate lines in each graph represent the different noise levels: $+ = -25\text{dB}$, $\circ = -30\text{dB}$, $\times = -35\text{dB}$, $\Box = -40\text{dB}$, $\ast = \text{no noise}$.
HSM can be constructed depending on the noise level and the desired percent error. The gratings shown here demonstrated a contrast ratio of above 70%, indicating that they will be adequate for use in constructing a preliminary version of the HSM.

If a further improvement in signal-to-noise ratio is desired, a heterodyne receiver configuration can be easily added to the holographic Stokesmeter architecture. This architecture is illustrated in Figure 5.5. The four diffracted beams represent the four intensities to be measured. These beams are mixed with a strong local oscillator using polarizing beam splitters. The local oscillator is chosen to be linearly polarized at 45 degrees so that both the perpendicular and the parallel components of the diffracted light will be mixed with the local oscillator evenly. Each of the eight beams is then sent through a standard heterodyne receiver architecture and the value of the perpendicular and parallel components of each of the four diffracted beams is recovered. These can then be used in conjunction with the measurement matrix to determine the four Stokes.

![Figure 5.5 Heterodyne receiver for Holographic Stokesmeter](image)
parameters. The heterodyne architecture has the advantage of helping to overcome the system noise and improve the signal-to-noise ratio by providing additional input intensity.

5.3 Measured Stokes parameters

In this section we demonstrate the complete operation of the HSM. We employed two sets of multiplexed gratings using rotation angles of $\gamma_1 = -4^\circ$ and $\gamma_2 = 4^\circ$. Each set was written at external angles of $40^\circ$ and $54^\circ$ for the first and second gratings respectively, with the reference beam at a $2^\circ$ angle. Figure 5.6 displays the diffraction efficiencies of the holographic samples [30]. It shows the polarization-sensitive diffraction efficiencies of 0.242, 0.211, 0.235, and 0.202 for the s-polarized state, and 0.147, 0.133, 0.148, and 0.113 for the p-polarized state. These diffraction efficiencies yield contrast ratios (CR) of 39.8%, 45.6%, 43.0%, and 45.5% respectively.

![Diffraction efficiencies vs. incident polarization of the two multiplexed HSM](image)

*Figure 5.6: Diffraction efficiencies vs. incident polarization of the two multiplexed HSM*
Here, CR is defined as the ratio of the difference between the diffraction efficiency of an s-polarized wave and that of a p-polarized wave to the diffraction efficiency of an s-polarized wave. Using the above parameters we constructed the measurement matrix for our system.

The measurement matrix relates the observed signals to the incident Stokes vector from equations (4.25) and (4.27)

\[
\begin{bmatrix}
I_{11} \\ I_{12} \\ I_{13} \\ I_{14}
\end{bmatrix} = \begin{bmatrix}
A_1 + B_1 & (A_1 - B_1)\cos(2\gamma_1) & (A_1 - B_1)\sin(2\gamma_1) & 0 \\
A_2 + B_2 & (A_2 - B_2)\cos(2\gamma_2) & (A_2 - B_2)\sin(2\gamma_2) & 0 \\
A_3 + B_3 & (A_3 - B_3)\cos(2\gamma_1) & 0 & -(A_3 - B_3)\sin(2\gamma_1) \\
A_4 + B_4 & (A_4 - B_4)\cos(2\gamma_2) & 0 & -(A_4 - B_4)\sin(2\gamma_2)
\end{bmatrix} \begin{bmatrix}
I \\ Q \\ U \\ V
\end{bmatrix}
\]

Here, (I, Q, U, V) represents the input Stokes parameters, and \(I_{ii}\) is the diffracted intensity from the i-th grating. \(A_i\) (\(B_i\)) characterizes the diffraction efficiencies of the i-th grating for s (p)-polarized input beams, taking into account the Fresnel reflections and transmissions at the interfaces. The angle \(\gamma_1\) (\(\gamma_2\)) denotes the rotation of the substrate containing grating 1 and 3 (2 and 4). In order for the matrix in (5.4) to be well-conditioned [31,32], the diffraction efficiencies of the four gratings need to be chosen properly along with the other parameters. Given that the coefficients of the matrix in (5.4) depend on polarization-sensitive diffraction it can be seen that in order to design a robust system, one must be able to control the amount of diffraction for each polarization very carefully. For example, it may be desirable for A–B to be a large negative quantity, which requires a grating that ideally diffracts only p-polarized light. One might also require A–B to be a large positive quantity, requiring a grating that ideally diffracts only s-polarized light.

We considered three different polarizations of the input beam: horizontal linear [1,1,0,0], linear [1,-0.55,0.84,0], and elliptical [1, 0.9, -0.25, -0.36]. These polarization states can be realized by passing the laser beam through a half-wave and a quarter-wave
plate rotated by appropriate angles. The intensity of each of the four diffracted beams was measured and used in the measurement matrix to determine the original Stokes vector.

For comparison, the values of the Stokes parameters describing the input polarization states were measured with a quarter-wave plate and a linear polarizer as shown in Figure 5.7. In this conventional method, we need to measure the four different intensities $I(0,0)$, $I(45,0)$, $I(45,90)$ and $I(90,0)$ by changing the angle of the linear polarizer, and inserting the quarter wave plate. Here, $\theta$ of $I(\theta,\Phi)$ is the angle of the linear polarizer, and $\Phi$ of $I(\theta,\Phi)$ is the quarter wave plate retardation. The output intensity for this method is given as [33]:

$$I_{\text{out}}(\theta,\Phi) = (1/2) \left[ I + Q \cos(2\theta) + U \cos(\Phi) \sin(2\theta) + V \sin(\Phi) \sin(2\theta) \right]$$  \hspace{1cm} (5.5)

From these intensity measurements we can calculate the Stokes parameters:

$$I = I(0,0) + I(90,0)$$

$$Q = I(0,0) - I(90,0)$$

$$U = 2I(45,0) - I(0,0) - I(90,0)$$

$$V = \frac{(2I(45,90) - I(0,0) - I(90,0))}{\eta}$$  \hspace{1cm} (5.6)

where $\eta$ is the absorption factor related to the quarter wave plate.

Figure 5.8 displays the measured average values and the standard deviations of $I$, $Q$, $U$ and $V$ for each method (the conventional method ‘x’ and the HSM method ‘○’).
Although the result for the HSM has a slightly larger error than the result for the QWP/linear polarizer, the average values are approximately equal to the assumed values within the error range. Note that the error in our conventional method is as high as 8%.

Figure 5.8 The average values and the standard deviations of I, Q, U and V measured by the three methods for the three different polarization states of the input beam (Quarter wave plate / Linear Polarizer: x, HSM: ○, new HSM: ●)
This implies that the reference optical elements (such as the polarizer, waveplates, etc.) contribute significantly to the error observed in the HSM data. More precise optical elements can be used to suppress these errors. In general, residual error in the HSM drops monotonically with increasing CR, which can be seen from an analysis of the robustness of this HSM [28]. The current gratings used in this HSM had CRs in the range of 40–50%. We expect that a HSM made with much higher CRs will produce more accurate results.

Chapter 6

Compact Architecture of a volume Holographic Stokesmeter

The current architecture of HSM needs two holographic substrates with 4 different imaging systems to construct a polarimetric image. In order to use a single imaging system we now describe a compact version of the HSM, and demonstrate the basic principle underlying it.

6.1 Basic operation mechanism

This architecture consists of a pair of spatially separated gratings in a single holographic substrate and two electro-optic modulators (EOM), as shown in Figure 6.1(a) [30]. The first EOM plays the role of the quarter-wave plate in the original architecture,
namely to interchange the U and V parameters. In the new architecture, it is necessary to have the ability to reverse the polarization states of the input beams (s-polarization into p-polarization and p-polarization into s-polarization). This can be accomplished by rotating the fast axis of the second EOM by 45° with respect to that of the first EOM. Figure 6.1(b) shows a possible sequence of driving voltages for the EOMs, which can be operated at very high (~GHz) frequencies. Since there is 0V sent to the EOMs during T1, the split input beams enter the HOE without changing the polarization state. Then, the measured intensities from the diffracted beams yield the information about the first two rows of the measurement matrix in equation (5.4). The intensity measurement for the reversed polarization states of the input beams after interchanging U and V parameters during T2 yields the information about the third and fourth rows of the measurement matrix in equation (5.4).
To implement the polarimetric imaging system using this new HSM, the output beams diffracted in parallel by the HOE are imaged onto two focal plane arrays (FPA) using a single 4f imaging system. After all four intensity measurements are taken during the periods of $T_1$ and $T_2$ for each pixel of the FPAs, we can construct the polarimetric image by manipulating the signals with precalibrated field programmable gate arrays (FPGA) for example. We have implemented the basic model of this new HSM by using a quarter-wave plate and a half-wave plate in place of the two EOMs. The Stokes parameters measured this way are also added in Figure 5.4 (new HSM ‘●’), and agree well with the previous methods, thus establishing the feasibility of this architecture.
This new HSM is structured to accommodate the ability to perform spectrally multiplexed polarimetry as well. Specifically, the single hologram in each zone of the substrate could be replaced by a set of angle multiplexed gratings. Each grating would be designed to produce a diffracted beam (orthogonal to the substrate) only for a specific band of frequencies at a specific angle of incidence. Thus, by scanning the angle of incidence without changing the position [34], it is possible to produce a polarimetric image for a desired wavelength band. For substrates with a thickness of 1mm, for example, the spectral width of each band is of the order of 0.5nm in the visible range, and the corresponding Bragg angular bandwidth is about 1 mrad. Thus, it may be possible to implement many such bands in a single device. The maximum number of bands would in practice be limited by the M/# of the material [23]. The spectral resolving function of this design will be discussed in section 6.3 in detail.

6.2 Demonstration of polarization imaging

6.2.1 Spatial Light Modulator (SLM)

A SLM is a device that converts incoherent images or digitized images into coherent optical information appropriate for optical processing. Since the output images of a SLM have different polarization states depending on the position in two-dimensional space, we can use this output image as an input of the HSM to demonstrate the polarization imaging.

Figure 6.2 show the cross section of a liquid crystal (LC) spatial light modulator [35]. Polarized light enters the device from the top, passes through cover glass, transparent electrode and liquid crystal layer, is reflected off the shiny pixel electrodes,
and returns on the same path. Electrical signals from the pins on the bottom of the pin-grid array package are transmitted to the VLSI die circuitry through the bond wires. The voltage induced on each electrode (pixel) produces an electric field between that electrode and the cover glass. This field produces a change in the optical properties of the LC layer, which are dependent on the type of LC material (Nematic Liquid Crystal or Ferroelectric Liquid Crystal).

Each of the 512 X 512 elements of the SLM can be programmed to 256 discrete voltage states on data driver IC during “on” signal of a gate bus line, resulting in amplitude modulation in the case of FLC, or phase modulation for NLC. These SLM devices do not exhibit a linear relationship between the modulation level and the pixel voltage because of the nonlinearity of LC’s response. This fact must be taken into account when displaying images and utilizing the SLM.

A FLC-based SLM can be though of as a half waveplate with an optic axis orientation that is voltage dependence. When linearity polarized input light is presented, the output light is likewise linearly polarized, depending on the rotation angle (0° ~ 180°)
of LC. When analyzed with a polarizer, the output is amplitude modulation along the real axis.

A NLC-based SLM is a variable birefringence device. As the voltage of the pixel changes, so does the retardance of the LC layer, producing a phase shift, or change in the polarization state. With no field present the device has its maximum birefringence, resulting in a maximum phase shift, usually $2\pi$. With the maximum field present the birefringence is at its minimum, through not at zero. Since NLC dose not respond to the polarity of the electric field, complimentary pixel values $n$ and $255-n$ result in the same phase shift.

### 6.2.2 Experimental Setup

The following figure 6.3 shows the measurement procedure to get polarimetric image using SLM images as an input. S-polarized light from a laser is reflected by the polarizing beam splitter (PBS) toward the SLM. The half waveplate (HWP1) then rotates light into the frame of the SLM’s optic axis. Light reflected from the SLM may or may not be modulated, depending on the voltage of the reflecting pixel. The reflected light from the normal beam splitter (BS) is reoriented back into the plane of the holographic element by the half wave plate (HWP2) and then is imaged onto the CCD camera using 4-f imaging systems. In this way, we are able vary the orientation of the linearly polarized light used to probe the grating. Then diffraction efficiencies for each pixel in the CCD array has been monitored for s- and p- polarization states, respectively, so that we can construct the 2-D array map of the measurement matrix for each pixel in the CCD array. Once we find this 2-D map of measurement matrix, we need to measure the four
diffracted intensities from the holographic gratings for each pixel. In order to use the compact version of the HSM we first measured two diffracted intensities ($I_1$ and $I_3$) from the two different gratings in a thick hologram as shown in figure 6.3(b), and then measured the other two diffracted intensities ($I_2$ and $I_4$) for the reversed polarization state of the input image using a quarter wave plate and half wave plate before the holographic element as shown in figure 6.3(c). By using the 2-D maps of measurement matrix and measured four diffracted intensities for each pixel in CCD, we can construct the polarimeric images from the SLM images. Note that since operating speed is not an issue for this proof-of-principle demonstration, we have implemented the HSM by using a quarter wave plate and a half wave plate instead of a pair of EOMs, which are necessary only for high speed operation.

Figure 6.4 shows the experimental setup to get polarization images using a conventional method, which consists of a quarter waveplate and a linear polarizer. Here, we used a HWP and a PBS instead of polarizer. First, we have measured the output intensities ($I(0,0)$, $I(45,0)$, $I(90,0)$) by simply rotating the fast axis of the HWP($\theta$) without quarter waveplate. Then the last intensity measurement $I(45,90)$ has been performed with rotating the fast axis of HWP by 45 degree and inserting the quarter waveplate. The measurement procedures using SLM image as an input for this conventional method are similar to the one by using the HSM except replacing the HOE elements with linear polarizer.
\[ \eta_p = \frac{I_p}{I_{op} + I_p}, \quad \eta_s = \frac{I_s}{I_{os} + I_s} \]

Figure 6.3 Experimental setup for the HSM system using SLM images as an input
(a) measurement procedure to get diffraction efficiencies for two different volume gratings
(b) intensity (\(I_{1t}\) and \(I_{2t}\)) measurement during \(T_1\) period
(c) intensity (\(I_{3t}\) and \(I_{4t}\)) measurement during \(T_2\) period
First, we used the simple input image consisting of p-polarization image in the left half and s-polarization image in the right half as in figure 6.4(a), and then we constructed the output image (Figure 6.4(b)) using the mentioned procedure for the HSM. And by rotating the polarization state of the input images with HWP for the same input image, we also constructed the different polarization signature as in figure 6.5(b). Even if there are some error points in the output images, the overall data is almost the same as the expected stoke parameters indicated red arrows.

**Figure 6.4** Experimental setup for the quarter waveplate/linear polarizer system using SLM images as an input:
(a) intensity measurement without quarter wave plate: I(0,0), I(45,0), I(90,0)
(b) intensity measurement with quarter wave plate: I(45,90)

### 6.2.3 Polarization Imaging

First, we used the simple input image consisting of p-polarization image in the left half and s-polarization image in the right half as in figure 6.4(a), and then we constructed the output image (Figure 6.4(b)) using the mentioned procedure for the HSM. And by rotating the polarization state of the input images with HWP for the same input image, we also constructed the different polarization signature as in figure 6.5(b). Even if there are some error points in the output images, the overall data is almost the same as the expected stoke parameters indicated red arrows.
Figure 6.5  Polarization imaging data 1 for horizontal and vertical polarization states
(a) SLM image as an input  (b) measured output image from the HSM
Figure 6.6 Polariation imaging data 2 for +45 degree and -45 degree polarization states
(a) SLM image as an input  (b) measured output image from the HSM
In order to demonstrate the feasibility of this imaging system for more realistic images, we used the check pattern (Fig. 6.7(c)) as an input image from the SLM. Here, white (Black) color represent the s (p) -polarization state. Before performing the measurement for the diffracted beams, we have to send the full white (Fig. 6.7(a)) and the full back pattern (Fig. 6.7(b)) onto the HOE because we need to find the measurement matrix in terms of diffracted amplitudes of planar and perpendicular polarization components.

![Figure 6.7 SLM images as a input](image)

(a) full white : s-polarization state (b) full balck : p-polarization state (c) check pattern : s – and p- polarization

By measuring the diffracted intensities from the HOE, we can construct the polarization image as shown in Figure 6.8 (a) for horizontally/vertically linear polarization state [1/1, 1/-1, 0/0, 0/0], (b) arbitrary linear polarization state [1/1, -0.3/0.14 ,0.94/0.95 , 0/0], and (c) the elliptical polarization state [1/1, 0.35/-0.2, 0.76/-0.3, 0.53/0.94], where [I,Q,U,V] represents average values of Stokes parameters. In case of linear polarization states, there are less errors than the elliptical case because the measurement errors resulting from the absorption or phase change due to misalignment of
the fast axis of QWP can be avoided. Thus, the image quality of the linear polarization case is better than that of the elliptical polarization case. Moreover, we observe that imaging data in Figure 6.8 has many error points compared the one in Figure 6.5 and 6.6, and the overall image quality is not good. There are some reasons for that. First, the input images from the SLM is not a perfect one because the pixilated nature of the SLM back plane produce diffraction, resulting in imaging only the zero-order component or one of the higher order. Thus, if we use high resolution images as an input, then the $0^{th}$ order or $1^{st}$ order diffracted beam intensity will be varied depending on the input image. Second, the error in our conventional method is still high as we can see form the figure 6.9 for

Figure 6.8 Polarization image measured by the HSM for different polarization state: (a) Horizontally /Vertically linear polarization,(b) arbitrary linear polarization , and (c) elliptical polarization

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Figure 3.9 Polarization image measured by the conventional method for different polarization state: (a) Horizontally /Vertically linear polarization, (b) arbitrary linear polarization, and (c) elliptical polarization.

three different case: [1/1, 1/-1, 0/0, 0/0], [1/1, -0.75/0.8, -0.54/0.4, 0/0], and [1/1, 0.7/0.3, -0.6/-0.42, 0.25/0.65]. This implies that the reference optical elements (such as the polarizer, waveplates, etc.) including the non-uniformity of the top surface of the SLM contribute significantly to the error observed in the HSM data. More precise optical elements can be used to suppress these errors. In addition, the holographic element should be uniform in space and have high contrast ratio of diffraction efficiency between s- and p- polarization state to reduce the residual error on the HSM.
6.3 A Spectrally Resolved Stokesmeter

The polarimetric signature is dependent upon several characteristics of the target, including the roughness of the target surface and the shape and orientation of the target as we discussed in chapter 1. On the other hand, spectral signature of an object depends on properties such as surface coatings or chemical compositions. These spectral signatures are the specific combination of reflected and absorbed electromagnetic (EM) radiation at varying wavelengths which can uniquely identify an object [36-38]. In general, polarization and multispectral imaging data are somewhat complementary because they depend on different optical characteristics of the object. Thus, it is in principle possible to perform contrast enhancement of a scene by using both polarization and spectral images of the scene.

The spectral resolving function of compact version of the HSM is achieved by multiplexing several gratings in each spatial location of the HOE. Each grating would be designed to produce a diffracted beam orthogonal to the substrate for a specific band of frequencies at a specific angle of incidence. In order to find the writing condition to produce the output diffracted beam orthogonal to the substrate, we briefly review the concept of holographic beam combiner [39], and then we derive the specific writing condition for our HSM [40].

Many holographic materials, though efficient in reading at many wavelengths, are particularly sensitive to a narrow band of wavelengths during writing. Therefore one may need to write and read holograms at different wavelengths. The following parameters are assumed to be given as inputs: the values of the read wavelengths $\lambda_{R1}$ and $\lambda_{R2}$, the write wavelength $\lambda_W$, the mean index of the holographic substrate, the thickness thereof, and
the amplitude of the index modulation for each grating. The equations are used to determine the following output parameters: the angles $\theta_{W1}$ and $\theta_{W2}$ (for both reference and object beams) used for writing the holograms, and the angles at which the lasers at $\lambda_{R1}$ and $\lambda_{R2}$ have to be applied to the beam combiner (i.e., the Bragg angles). The desired output angle $\theta_S$ of the combined beam can be treated as a given input or as an output to be determined depending on the desired degree of crosstalk suppression. This is because the value of $\Delta \lambda_{\text{FWHM}}$ depends on this angle, in addition to other parameters that are assumed to be given [22]. In the following discussion we show the relationship between the write angles and read angles when different wavelength lasers are employed for each step.

Figure 6.10 illustrates the basic writing geometry. Consider the process for writing the first grating, using beams W1 (reference) and W2 (object) with laser beams of wavelength $\lambda_W$ (e.g. 514.5 nm). We choose the first read wavelength $\lambda_{R1}$ (e.g. 980 nm) and the desired angle of diffraction $\theta_S$. These two choices plus the chosen writing wavelength will determine the first two writing angles $\theta_{W1}$ and $\theta_{W2}$. If read by a laser beam also at $\lambda_W$, the read beam will diffract efficiently only if it is Bragg matched, i.e.,

![Writing Geometry](image)

*Figure 6.10 Schematic illustration of the geometry for writing two holograms at 514.5nm.*
incident at the same angle as W2, and will produce a diffracted beam on the other side parallel to the reference beam W1. When read by laser beam O1 at $\lambda_{R1}$ the Bragg incidence angle as well as the diffracted angle $\theta_S$ is larger. Consider next the process for writing the second grating, using a new pair of beams with $\lambda_w$ (W1’ and W2’ from Figure 6.10). Our goal is to choose the incident angles for these two beams such that when this hologram is read by a laser beam O2 at the second read wavelength of $\lambda_{R2}$ the diffracted beam will emerge at the same angle $\theta_S$. We choose the angular distance between the first and the second read beams $\delta$, as well as the wavelength of the second read beam, O2. These constraints yield a new pair of writing angles, $\theta'_w1$ and $\theta'_w2$, for the beams W’1 and W’2, respectively, in Figure 6.10. Explicit analysis shows that these two angles are given by

\[
\theta'_w1 = \sin^{-1} \left[ n_w \cdot \sin \left( \sin^{-1} \left( \frac{n_R}{n_w} \cdot \frac{\lambda_w}{\lambda_R} \cdot \sin \left( \frac{\bar{\theta}_S + \bar{\delta}}{2} \right) \right) - \bar{\delta}/2 \right) \right],
\]

\[
\theta'_w2 = \sin^{-1} \left[ n_w \cdot \sin \left( \sin^{-1} \left( \frac{n_R}{n_w} \cdot \frac{\lambda_w}{\lambda_R} \cdot \sin \left( \frac{\bar{\theta}_S + \bar{\delta}}{2} \right) \right) + \bar{\delta}/2 \right) \right],
\]

where we defined

\[
\bar{\theta}_S = \sin^{-1} \left( \frac{\sin \theta_S}{n_R} \right), \quad \bar{\delta} = \sin^{-1} \left( \frac{\sin (\theta_S + \delta)}{n_R} \right) - \sin^{-1} \left( \frac{\sin \theta_S}{n_R} \right),
\]

\[
\delta = \theta_{Oj} - \theta_{O(j+1)}
\]

$n_w \equiv \text{index at the writing wavelength}, \quad \lambda_w \equiv \text{the writing wavelength},
\]

$n_R \equiv \text{index at reading wavelength}, \quad \lambda_R \equiv \text{the reading wavelength}.$

Now, we consider the geometry of an HSM during the writing step in figures 6.8(a) and (b). In this figure, $n_1$ and $n_2$ are the indices of refraction outside and inside the
substrate, respectively. For one of the gratings (Fig. 6.11(a)), \(2\theta_0\) is the angle between the two writing beams. \(2\theta_0\)' shows the corresponding angle inside the substrate. Similarly, for the other gratings (Fig. 6.11(b)), \(2\theta_1\) is the angle between the writing beams, and \(2\theta_1\)' is the corresponding angle inside the substrate. Both gratings are written with light field of a single color \((\lambda_G)\). From the Bragg condition for a thick hologram, the grating periods for each grating are given by

\[
\Lambda_o = \frac{\lambda_G}{2 \sin \theta_0} = \frac{\lambda_G}{2 \sin \left( \frac{1}{2} \sin^{-1} \left( \frac{n_1}{n_2} \sin 2\theta_0 \right) \right)} \quad (6.3)
\]

\[
\Lambda_1 = \frac{\lambda_G}{2 \sin \theta_1} = \frac{\lambda_G}{2 \sin \left( \frac{1}{2} \sin^{-1} \left( \frac{n_1}{n_2} \sin (2\theta_1 + \delta) \right) - \frac{1}{2} \sin^{-1} \left( \frac{n_1}{n_2} \sin \delta \right) \right)} \quad (6.4)
\]

In the reading step, an input beam at \(\lambda_G\) incident at angle \(2\theta_0\) will produce a diffracted beam perpendicular to the substrate for the first grating, as shown in Figure 6.11(c). For the second grating, to be read at a different wavelength \((\lambda_R)\), we can ensure that the diffracted beam will be perpendicular to the substrate by choosing the proper value of \(2\theta_1\) and \(\delta\). The grating period of the second grating will have the following relations for two different wavelengths to satisfy the Bragg condition as shown in Figure 6.11(d)

\[
\Lambda_1 = \frac{\lambda_G}{2 \sin \theta_1} = \frac{\lambda_R}{2 \sin \theta_R} \quad (6.5)
\]

or

\[
\frac{\lambda_G}{2 \sin \left( \frac{1}{2} \sin^{-1} \left( \frac{n_1}{n_2} \sin (2\theta_1 + \delta) \right) - \frac{1}{2} \sin^{-1} \left( \frac{n_1}{n_2} \sin \delta \right) \right)} = \frac{\lambda_R}{2 \sin \left( \frac{1}{2} \sin^{-1} \left( \frac{n_1}{n_2} \sin 2\theta_R \right) \right)} \quad (6.6)
\]

where \(2\theta_R\) is equal to \(2\theta_1 + \delta\) for this geometry. Therefore, an input beam at \(\lambda_R\) incident
\[2\theta_R = \frac{n_2}{n_1} \sin \left(2 \sin^{-1}\left[\frac{\lambda_R}{\lambda_G} \sin \left(\frac{1}{2} \sin^{-1}\left(\frac{n_1}{n_2} \sin(2\theta_1 + \delta)\right) - \frac{1}{2} \sin^{-1}\left(\frac{n_1}{n_2} \sin \delta\right)\right)\right]\right)\]

(6.7)

**Figure 6.11**
(a) Writing condition with Nd-Yag laser beam (532nm) for grating 1  
(b) Writing condition with Nd-Yag laser beam (532nm) for grating 2  
(c) Reading condition with Nd-Yag laser beam (532nm) for grating 1  
(d) Reading condition with He-Ni laser beam (633nm) for grating 2  

where \(\theta_R = 2\theta_1 + \delta\)  

\((\Lambda_0, \Lambda_1: the grating periods, G: Nd-Yag laser, R: He-Ni laser)\)

at angle \(2\theta_R\) for the 2\textsuperscript{nd} grating will produce a diffracted beam orthogonal to the substrate.

Using this method, we can create many angle-multiplexed gratings at the same location in a HOE of our compact device, where the maximum number of gratings would be limited by the M/# of the material [23]. Thus, by scanning the angle of \(\theta_R\) without changing the
location [34], it is possible to produce a spectrally resolved polarimetric image for a desired wavelength band.

In order to demonstrate the feasibility of the spectrally multiplexed Stokesmeter, we wrote two angle-multiplexed gratings in both spatial positions of a 2mm thick Memplex®[24] sample with a 532nm frequency doubled Nd:YAG laser using the rotation angles of \( \gamma_1 = 0^\circ \) and \( \gamma_2 = 4^\circ \) for the substrate. To simplify the analysis, we chose the same writing angles of \( 2\theta_0 \) and \( 2\theta_1 \approx 38^\circ \) for each grating, and a rotation angle of \( \delta \approx 12.2^\circ \) from Eq.(6.5) for the writing condition. That is, each set was written at external angles of \( 38^\circ \) and \( 50.2^\circ \) with the reference beam normal to the substrate for the first and second gratings, respectively. By scanning the angle of incidence between \( 38^\circ \) and \( 50.2^\circ \) for wavelength of 532nm and 633nm, the output diffracted beams orthogonal to the HOE have been produced.

Figure 6.12 shows the measured diffraction efficiencies from the two gratings in a

![Figure 6.12 Diffraction efficiencies vs. incident polarization of the two angle-multiplexed gratings, shown for four different contrast ratios (CR)](image)
thick hologram for two different wavelengths of 532nm and 633nm. These diffraction efficiencies yield contrast ratios of 53.3% and 49.5% for 532nm, and of 71.9% and 74.7% for 632nm at two spatially separated locations. Based on this data, we can construct the measurement matrix of Eq.(5.4) for each wavelength. Here, we considered three different polarizations of the input beam: horizontal linear [1, 1, 0, 0], linear [1, 0.94, 0.35, 0], and elliptical [1, 0.94, -0.17, 0.30]. Then, we confirm the basic feasibility of this HSM in the slow-speed regime (a pair of EOMs for high speed operation, or wave plates for slow-speed operation). Figure 6.13 displays the measured average values and the standard deviations of I, Q, U and V using this compact architecture for each wavelength (532nm, 633nm). The average values for each wavelength are approximately equal to the assumed values within the error range. This shows the feasibility of a spectrally multiplexed HSM in its compact architecture. One of the interesting applications for this device will be real time filtered-camera system such as a Stokes Parameter Camera, where after attaching the HOE in front of the lens of any camera system, the spectrally resolved image can be produced by simply rotating the HOE with a specific angle corresponding to a specific wavelength.
Figure 6.13 The average values and the standard deviations of I, Q, U and V measured by the Spectrally scanned HSM for three different polarization states of the input beams.
Chapter 7

Photonic Crystal

Since photonic band gap (PBG) materials have spatially periodic dielectric functions, these materials can be viewed as a two-or three-dimensional extensions of Bragg mirrors. Holograms also have spatially varying dielectric constants that perform the complete reconstruction of the electromagnetic field as we discussed previous chapters. In this respect, PBGs can be considered as a limit of volume holographic mirrors with a high-contrast periodic modulation of their complex refractive index. In this chapter, we will briefly describe general properties of a PBG structure and FDTD simulation methods, and then illustrate some fabrication examples to implement a photonic crystal.

7.1 Photonic Band Gap (PBG)

A crystal is a periodic arrangement of an identical unit structure (such as atoms or molecules) in the dimensions of space. A crystal, therefore presents of a periodic potential to an electron propagating through it. This periodic potential modifies the energy–momentum relation of electrons due to Bragg–like diffraction from the different lattice planes as shown in Figure 7.1. The Bragg condition reads $2d \sin(\theta) = m\lambda$ and determines mirror–like behavior of crystal planes of distance $d$ for an oblique incident plane wave with angle $\theta$ parallel to the plane. It only occurs for wavelengths $\lambda \leq 2d$. If for example the incident wave vector $k = 2\pi/\lambda$ becomes $\pi/d$ at $\theta = 90^\circ$ it is reflected, i.e. encounters a forbidden frequency for which it cannot propagation. Therefore electrons
are forbidden to propagate with certain energies in certain directions. If the lattice potential is strong enough, the gap might extend to all possible direction, resulting in a complete band gap. For example, a semiconductor has a complete band gap between the valance band and the conduction band.

Closely related to the electronic properties are also the optical properties. The periodic potential in a photonic crystal is due to a lattice of macroscopic dielectric media instead of atoms. If the dielectric constants of the materials in the crystal are different enough, and the absorption of light by the material is minimal, then scattering at the interfaces can produce many of the same phenomena for photons as the atomic potential does for electrons. The appearance of the band gaps is highly dependent on the type of lattice and on the ratio of the periodic element to background permittivity. Intentionally introduced defects in the crystal allow localization of light in a confined state (linear waveguides and point-like cavities) just like electronic dopants in solid state crystals.

Since there is a strong analogy between the concepts of solid state physics and the artificial electromagnetic crystal structures, a photon in a photonic crystal can be treated in the same way like an electron in a semiconductor crystal. The study of photonic
crystals is likewise governed by the Bloch-Floquet theorem [40], which tells us that, for a Hermitian eigenproblem whose operator are periodic functions of position, the solution always can be chosen of the periodic function (~ e^{ikx}). The dispersion relation of a photonic crystal can be described with band structures, the reciprocal lattice and the Brillouin zone concept [41].

The Schröinger equation for an electron of effective mass \( m \) in a crystal, in which the potential is \( V(r) \), can be written as:

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \Psi(r) = E \Psi(r)
\]

(7.1)

where \( V(r) \) is a periodic function with the periodicity of the lattice, \( R \):

\[ V(r) = V(r + R) \]

(7.2)

The eigenstates of this equation are also periodic functions with period \( R \). The dispersion relationship derived, \( E(k) \), will present a forbidden band for all energies \( E \) which have imaginary values. Similarly, in a medium in which a spatial modulation of the dielectric constant \( \varepsilon(r) \) exists, photon propagation is governed by the classical wave equation for the magnetic field \( H(r) \):

\[
\nabla \times \left[ \frac{1}{\varepsilon(r)} \nabla \times H(r) \right] = \left( \frac{\omega}{c} \right)^2 H(r)
\]

(7.3)

In a photonic crystal, \( \varepsilon(r) \) is a periodic function:
These equations show the parallelism between electrons in crystalline solids and photons in a photonic crystal. On the other hand, there are some differences between electron waves in a semiconductor and electromagnetic waves in a photonic crystal. In a photonic crystal, the non-negative energy eigenvalues occur in the electromagnetic equations because of the squared values of the eigenfrequencies. Also, for a semiconductor there is an importance for the transitions between the different energy bands and levels, whereas for a photonic crystal the existence or absence of electromagnetic modes at certain frequencies is main interest.

7.1.1 Maxwell’s equation in Period Media

In a macroscopic medium, including photonic crystal, the four macroscopic Maxwell equations in cgs unit are

\[ \nabla \cdot B = 0 \quad \nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0 \]

\[ \nabla \cdot D = 4\pi \rho \quad \nabla \times H - \frac{1}{c} \frac{\partial D}{\partial t} = \frac{4\pi}{c} J \]

(7.5)

(7.6)

where \( E \) and \( H \) respectively denote the macroscopic electric and magnetic fields’ strengths, \( D \) and \( B \) denote the dielectric displacement and magnetic induction fields’ strengths, \( \rho \) and \( J \) are the densities of free charges and currents, and \( c \) is the speed of light in free space.
The propagation of electromagnetic waves is assumed to be in a mixed dielectric medium consisting of macroscopic homogeneous domains of different dielectric materials with no free charges or currents \((\rho = 0 \text{ and } J = 0)\). Generally, the components of the dielectric displacement field \(D\) are related to the components of the electric field \(E\) via a power series [42]:

\[
D_i = \sum_j e_{ij} E_j + \sum_{j,k} \phi_{ijk} E_j E_k + O(E^3)
\]

(7.7)

For the most dielectric materials the following assumptions can be employed. First, the field strengths are small enough for linear approximation, i.e. the coefficients of all terms higher than first order in \(E\) vanish. Second, the material is macroscopic and isotropic, therefore \(E(r,t)\) and \(D(r,t)\) are related by a scalar dielectric constant \(\varepsilon(r,\omega)\). For the anisotropic case, an analog formalism for \(\mu\) can be applied. Third, \(\varepsilon(r,\omega)\) does not explicitly depend on the frequency \(\omega\) of the electromagnetic wave. Fourth, the medium is intended to be lossless, which means \(\varepsilon(r)\) can be taken as a real number. Then, we have \(D(r,t) = \varepsilon(r)E(r,t)\). For most dielectric materials of interest, the magnetic permeability \(\mu(r)\) is close to unity, so that we may set \(B(r, t) = \mu(r)H(r, t)\). Taking all these simplifying assumptions into account, for a macroscopic medium, whose physical properties can be assumed to be linear and isotropic, the Maxwell equations become

\[
\nabla \cdot H(r,t) = 0 \quad \nabla \times E(r,t) + \frac{1}{c} \frac{\partial H(r,t)}{\partial t} = 0
\]

(7.8)

\[
\nabla \cdot (\varepsilon(r)E(r,t)) = 0 \quad \nabla \times H(r,t) - \frac{\varepsilon(r)}{c} \frac{\partial E(r,t)}{\partial t} = 0
\]

(7.9)
Since the equations (7.8) and (7.9) are linear in $E$ and $H$, the time dependency can be separated out by expanding into a set of harmonic modes. In fact, this is a restriction of the fields to vary harmonically (in terms of a sinusoidal pattern) in time, but any solution can be constructed by a Fourier transformation of an adequate combination

Since only the real parts of the complex–valued fields correspond to the physical fields, we can write a harmonic mode as a certain field pattern time complex potential:

$$E(r,t) = E(r) \cdot e^{j\omega t} \quad (7.10)$$

$$H(r,t) = H(r) \cdot e^{j\omega t} \quad (7.11)$$

The combination of equations (7.10) and (7.11) in the divergence equations of (7.5) and (7.6) leads to the solenoidal conditions for the fields

$$\nabla \cdot H(r) = 0$$

(7.12)

$$\nabla \cdot D(r) = 0$$

(7.13)

There are no point–like sources or drains within the medium, so that the electromagnetic waves have to be transverse. Inserting (7.10) and (7.11) in both of the two curl equations from (7.8) and (7.9) yields

$$\nabla \times E(r) + \frac{i\omega}{c} H(r) = 0$$

(7.14)

$$\nabla \times H(r) - \frac{i\omega}{c} \varepsilon(r) E(r) = 0$$

(7.15)
The equations (7.14) and (7.15) can be decoupled by taking the curl and combining them with each other. As a result, the corresponding wave equations for the magnetic and electric eigenmodes are:

\[
\nabla \times \left( \frac{\nabla \times H(r)}{\varepsilon(r)} \right) = \left( \frac{\omega}{c} \right)^2 H(r)
\]

(7.16)

\[
\nabla \times (\nabla \times E(r)) = \left( \frac{\omega}{c} \right)^2 \varepsilon(r) E(r)
\]

(7.17)

where Eqs. (7.16) and (7.17) are called master equation. From the rearrangement of the equations (7.14) and (7.15), the electric field \( E(r) \) can be evaluated:

\[
E(r) = -\frac{ic}{\omega \varepsilon(r)} \left( \nabla \times H(r) \right)
\]

(7.18)

\[
H(r) = \frac{ic}{\omega} (\nabla \times E(r))
\]

(7.19)

The valid modes have to satisfy the condition (7.12) and (7.13). In analogy, this can be done in the same manner for the electric field \( E(r) \) with the condition (7.12) and (7.13). As we can see the above equations, if we perform a series of operations on a function \( H(r) \), and \( H(r) \) is really an allowable electromagnetic mode, then the result will be a constant time the original function \( H(r) \). That is, this is a eigenvalue problem. Thus, we identify the left side of the master equation as an operator \( \Theta \), acting on \( H(r) \), and equations (7.16) and (7.17) can be written as
The differential operator defined in the above equations is linear, i.e. any linear combination of solutions is itself a solution. The eigenvectors $E(r)$ are the field pattern according to the harmonic modes and the eigenvalues $(\omega/c)^2$ are proportional to the corresponding squared frequencies of those modes. Furthermore the operator is Hermitian under the standard inner product. For real $\varepsilon(r) > 0$ it is positive–definite what implies real eigenfrequencies $\omega$. Inserting (7.7) into equation (7.17) yields

$$\nabla \times \left( \nabla \times \frac{D(r)}{\varepsilon(r)} \right) = \left( \frac{\omega}{c} \right)^2 D(r)$$

(7.22)

And in analogy to (7.22)

$$\Xi D(r) = \left( \frac{\omega}{c} \right)^2 D(r)$$

(7.23)

$$\Xi D(r) = \nabla \times \left( \nabla \times \frac{D(r)}{\varepsilon(r)} \right)$$

(7.24)

an operator $\Xi$ can be defined. $\Xi$ is not Hermitian, because of the misplaced $\varepsilon(r)$ compared to the operator (7.22). Multiplying (7.24) by $1/\varepsilon(r)$ allows to define a generalized
eigenvalue equation, which is a more complicated numerical task. Defining a new field \( F(r) \)

\[
F(r) = \frac{1}{\sqrt{\varepsilon(r)}} D(r)
\]

(7.25)

would bring the generalized eigenvalue equation back in the form of a simple eigenvalue equation

\[
\frac{1}{\sqrt{\varepsilon(r)}} \nabla \times \left( \nabla \times \frac{1}{\varepsilon(r)} \right) F(r) = \left( \frac{\omega}{c} \right)^2 F(r)
\]

(7.26)

including a Hermitian differential operator, but anyway the field \( F(r) \) is not transverse. Obviously, there is no adequate efficient way to fix this problem without imposing more difficulties.

### 7.1.2 Brillouin Zones and the Origin of Photonic Band Gap

**Bloch waves and Brillouin zones**

A photonic crystal corresponds to a periodic dielectric function \( \varepsilon(x) = \varepsilon(x + R_i) \)

for some primitive lattice vectors \( R_i \) (\( i = 1, 2, 3 \) for a crystal periodic in all three dimensions). In this case, the Bloch-Floquet theorem for periodic eigenproblems states that the solutions to Eq. (7.16) can be chosen of the form \( \boldsymbol{H}(x) = e^{i k \cdot x} \boldsymbol{H}_{n,k}(x) \) with eigenvalues \( \omega_n(k) \), where \( \boldsymbol{H}_{n,k} \) is a periodic envelope function satisfying:
yielding a different Hermitian eigenproblem over the primitive cell of the lattice at each Bloch wavevector \( \mathbf{k} \). This primitive cell is a finite domain if the structure is periodic in all directions, leading to discrete eigenvalues labelled by \( n = 1, 2, 3, \cdots \). These eigenvalues \( \omega_n(\mathbf{k}) \) are continuous functions of \( \mathbf{k} \), forming discrete “bands” when plotted versus the latter, in a “band structure” or dispersion diagram—both \( \omega \) and \( \mathbf{k} \) are conserved quantities, meaning that a band diagram maps out all possible interactions in the system. (Note also that \( \mathbf{k} \) is not required to be real; complex \( \mathbf{k} \) gives evanescent modes that can exponentially decay from the boundaries of a finite crystal, but which cannot exist in the bulk) Moreover, the eigensolutions are periodic functions of \( \mathbf{k} \) as well: the solution at \( \mathbf{k} \) is the same as the solution at \( \mathbf{k} + \mathbf{G}_j \), where \( \mathbf{G}_j \) is a primitive reciprocal lattice vector defined by \( \mathbf{R}_i \cdot \mathbf{G}_j = 2\pi \delta_{i,j} \). Thanks to this periodicity, one need only compute the eigensolutions for \( \mathbf{k} \) within the primitive cell of this reciprocal lattice—or, more conventionally, one considers the set of inequivalent wavevectors closest to the \( \mathbf{k} = 0 \) origin, a region called the first Brillouin zone. For example, in a one-dimensional system, where \( R_1 = a \) for some periodicity \( a \) and \( \mathbf{G}_1 = 2\pi/a \), the first Brillouin zone is the region \( k = -\pi/a \cdots \pi/a \); all other wavevectors are equivalent to some point in this zone under translation by a multiple of \( \mathbf{G}_1 \). Furthermore, the first Brillouin zone may itself be redundant if the crystal possesses additional symmetries such as mirror planes; by eliminating these redundant regions, one obtains the irreducible Brillouin zone, a convex polyhedron that can be found tabulated for most crystalline structures. In the preceding one-dimensional
example, since most systems will have time-reversal symmetry \((k \rightarrow -k)\), the irreducible Brillouin zone would be \(k = 0 \cdots \pi/a\).

The familiar dispersion relations of uniform waveguides arise as a special case of the Bloch formalism: such translational symmetry corresponds to a period \(a \rightarrow 0\). In this case, the Brillouin zone of the wavevector \(k\) (also called \(\beta\)) is unbounded, and the envelope function \(H_{n,k}\) is a function only of the transverse coordinates.

**The origin of the photonic band gap**

A complete photonic band gap is a range of \(\omega\) in which there are no propagating (real \(k\)) solutions of Maxwell’s equations for any \(k\), surrounded by propagating states above and below the gap. There are also incomplete gaps, which only exist over a subset of all possible wavevectors, polarizations, and/or symmetries. We discuss both sorts of gaps in the subsequent sections, but in either case their origins are the same, and can be understood by examining the consequences of periodicity for a simple one-dimensional system.

Consider a one-dimensional system with uniform \(\varepsilon = 1\), which has planewave eigensolutions \(\omega(k) = ck\). This \(\varepsilon\) has trivial periodicity \(a\) for any \(a \geq 0\), with \(a = 0\) giving the usual unbounded dispersion relation. We are free, however, to label the states in terms of Bloch envelope functions and wavevectors for some \(a \neq 0\), in which case the bands for \(|k| > \pi/a\) are translated (“folded”) into the first Brillouin zone, as shown by the dashed lines in Figure 7.2(left). In particular, the \(k = -\pi/a\) mode in this description now lies at an equivalent wavevector to the \(k = \pi/a\) mode, and at the same frequency; this accidental
degeneracy is an artifact of the “artificial” period we have chosen. Instead of writing these wave solutions with electric fields $E(x) \sim e^{\pm i\pi x/a}$, we can equivalently write linear

combinations $e(x) = \cos(\pi x/a)$ and $o(x) = \sin(\pi x/a)$ as shown in Figure 7.3, both at $\omega = c\pi/a$. Now, however, suppose that we perturb $\varepsilon$ so that it is nontrivially periodic with period $a$; for example, a sinusoid $\varepsilon(x) = 1 + \Delta \cos(2\pi x/a)$, or a square wave as in the inset of Figure 7.2. In the presence of such an oscillating “potential,” the accidental degeneracy between $e(x)$ and $o(x)$ is broken: supposing $\Delta > 0$, then the field $e(x)$ is more concentrated in the higher-$\varepsilon$ regions than $o(x)$, and so lies at a lower frequency. This opposite shifting of the bands creates a band gap, as depicted in Figure 7.2(right). (In fact, from the

![Figure 7.2](image)

**Figure 7.2** (a) Dispersion relation (band diagram), frequency versus wave vector $k$, of a uniform one-dimensional medium, where the dashed lines show the “folding” effect of applying Bloch’s theorem with an artificial periodicity $a$. (b) Schematic effect on the bands of a physical periodic dielectric variation (inset), where a gap has been opened by splitting the degeneracy at the $k = \pm \pi/a$ Brillouin-zone boundaries (as well as a higher-order gap at $k = 0$).
perturbation theory described subsequently, one can show that for $\Delta << 1$ the band gap, as a fraction of mid-gap frequency, is $\Delta \omega / \omega \approx \Delta / 2$). By the same arguments, it follows that any periodic dielectric variation in one dimension will lead to a band gap, albeit a small gap for a small variation. More generally, it follows immediately from the properties of Hermitian eigensystems that the eigenvalues minimize a variational problem:

$$\omega_{n,k}^2 = \min_{E_{n,k}} \left( \frac{\iint |(\vec{\nabla} + i k) \times \vec{E}_{n,k}|^2}{\iint \varepsilon |\vec{E}_{n,k}|^2} \right) \epsilon^2,$$

(7.28)

in terms of the periodic electric field envelope $E_{n,k}$ where the numerator minimizes the “kinetic energy” and the denominator minimizes the “potential energy.” Here, the $n > 1$ bands are additionally constrained to be orthogonal to the lower bands:
for \( m < n \). Thus, at each \( \mathbf{k} \), there will be a gap between the lower “dielectric” bands concentrated in the high dielectric (low potential) and the upper “air” bands that are less concentrated in the high dielectric: the air bands are forced out by the orthogonality condition, or otherwise must have fast oscillations that increase their kinetic energy. (The dielectric/air bands are analogous to the valence/conduction bands in a semiconductor.)

In order for a complete band gap to arise in two or three dimensions, two additional hurdles must be overcome. First, although in each symmetry direction of the crystal (and each \( \mathbf{k} \) point) there will be a band gap by the one-dimensional argument, these band gaps will not necessarily overlap in frequency (or even lie between the same bands). In order that they overlap, the gaps must be sufficiently large, which implies a minimum \( \varepsilon \) contrast (typically at least 4/1 in 3d). Since the 1d mid-gap frequency \( \sim c\pi/a\sqrt{\varepsilon} \) varies inversely with the period \( a \), it is also helpful if the periodicity is nearly the same in different directions—thus, the largest gaps typically arise for hexagonal lattices in 2D and fcc lattices in 3D, which have the most nearly circular/spherical Brillouin zones. Second, one must take into account the vectorial boundary conditions on the electric field: moving across a dielectric boundary from \( \varepsilon \) to some \( \varepsilon' < \varepsilon \), the inverse “potential” \( \varepsilon|\mathbf{E}|^2 \) will decrease discontinuously if \( \mathbf{E} \) is parallel to the interface (\( \mathbf{E}_\parallel \) is continuous) and will increase discontinuously if \( \mathbf{E} \) is perpendicular to the interface (\( \varepsilon\mathbf{E}_\perp \) is continuous). This means that, whenever the electric field lines cross a dielectric boundary, it is much harder to strongly contain the field energy within the high dielectric, and the converse is true when the field lines are parallel to a boundary. Thus, in order to
obtain a large band gap, a dielectric structure should consist of thin, continuous veins/membranes along which the electric field lines can run—this way, the lowest band(s) can be strongly confined, while the upper bands are forced to a much higher frequency because the thin veins cannot support multiple modes (except for two orthogonal polarizations). The veins must also run in all directions, so that this confinement can occur for all \( k \) and polarizations, necessitating a complex topology in the crystal.

### 7.1.3 Dimensionality

#### 1-D Photonic Crystal

Figure 7.4 [43] shows the simplest possible photonic crystal in one dimension, which consists of alternating layers of materials with different dielectric constants. This photonic crystal can act as a perfect mirror for light with a frequency within a sharply defined gap, and can localize light modes if there are any defects in its structure. The 1D structure has wide applications such as the optical switch, the photonic band-edge laser, and the multiple-wavelength filter [44-46].

![Figure 7.4 The multilayer film: a one-dimensional photonic crystal](image)

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2-D Photonic Crystal

When the crystal is periodic in two directions and homogeneous in the third, photonic band gap appear in the plane of periodicity. Figure 7.5 shows a typical specimen, consisting of a square lattice of dielectric columns [43]. For certain values of the column spacing, this crystal have a photonic band gap in the xy-plane. Inside this gap, no extended state is permitted, and incident light is reflected. But although the multilayer film in 1D case only reflects light at normal incidence, this two-dimensional photonic crystal can reflect light incident from any direction in the plane. For light propagating in this plane, we can separate the modes into two independent polarizations (TE and TM mode), which will have different band gap structure of each. By combining the two band structures, we can design a photonic crystal that has band gaps for both polarizations. That is, by adjusting the dimensions of the lattice, we can arrange for the band gap overlap, resulting in complete band gap for all polarization. Furthermore, by perturbing a

![Image](image_url)

*Figure 7.5 Two-dimensional photonic crystal: square lattice with radius r and dielectric constant ε. The material is homogeneous in z-direction*
single lattice site, we can permit a single (localized) mode or set of closely spaced modes that have frequencies in the gap. For example, we can remove a single column from the crystal, or replace it with another whose size, shape, or dielectric constant is different from the original. Removing one column may introduce a peak into the crystal’s density of states. If the peak happens to be located in the photonic band gap, then the defect-induced state must be evanescent. Defect mode cannot penetrate the rest of the crystal, since it has a frequency in the band gap. A line defect, conversely, is a defect that extends periodically with infinite extent in one axial direction (usually a symmetry direction of the crystal) and with finite extent in the lateral directions. Because of its one-dimensional periodicity, a line defect forms a waveguide, and introduces a guided mode band(s) that has a one-dimensional Bloch wavevector and which is localized in the lateral directions. The Bloch-Floquet magic implies that such waveguide modes do not scatter—unless additional defects, or fabrication disorder, are introduced. Unlike traditional index-guided waveguides, however, a photonic-crystal waveguide cannot radiate laterally even in the presence of disorder or additional defects—light can only be scattered forwards or backwards—and it thus forms an effectively one-dimensional system in which many simple resonant phenomena can be explored.

The most prominent examples of this 2D photonic crystal are photonic crystal fibers and its applications [47, 48]. Photonic crystal slabs are built from a layered structure that consists of a high refractive index film sandwiched inside a low refractive index medium therefore confining the light normal to the slab plane by total internal reflection (TIR) [49]. The 2D slab structure offers the advantage of an easy process compared to 3D PBG fabrication.
3-D Photonic Crystal

Three dimensional photonic crystals such as Figure 7.6 [43] can have the novel properties including band gaps, defect modes, and surface states. Inserting a defects still allow us to localized light in a plane or on a line, but in three dimensions we have the additional freedom to create guided linear modes and modes that are localized at a single point. For this 3D case, complete band gap are rare. The gap must smoother the entire three-dimensional Brillouin zone, not just any one plane or line. However, Ho, Chan, and Soukoulis [50,51] found that a complete photonic band gap exists whether one embeds dielectric spheres in air or air spheres in a dielectric medium, as long as the sphere radius is chosen appropriately. In this 3D photonic crystal, we can perturb a single lattice site and thereby trap light at a single point in the crystal. With the point defect, we pull a state
from the continuum above or below the gap into the gap itself, and a localized mode results. Line defects such as Yabronovite [52] might be considered as a linear array of point defects. By choosing a proper radius and orientation for a line defect, it is possible to create the defect band with frequencies in the photonic band gap of the crystal. The state in this band extends along the defect, but decay exponentially into the rest of the crystal. Electromagnetic surface mode, which decays away exponentially on both side of the surface plane varies as the surface termination changes. The first 3D photonic crystal was used for the micrometer wavelength application, and many different structures in 2D and 3D have been developed [53].

7.1.4 Scaling Property

As we can anticipate the result of the calculus, there is no fundamental length scale for electromagnetism in dielectric media including photonic crystals. Now we suppose a medium whose dielectric constant $\varepsilon(r)$ is scaled by a parameter $s$: $\varepsilon'(r) = \varepsilon(r/s)$. Then by changing the variables in (7.16), using $r' = sr$ and $\Delta' = \Delta/s$:

$$s\nabla' \times \left( \frac{1}{\sqrt{\varepsilon'(r')/s}} s\nabla' \times H(r'/s) \right) = \left( \frac{\omega}{c} \right)^2 H(r'/s)$$

(7.30)

But, $\varepsilon'(r'/s) = \varepsilon'(r')$ and after dividing the equation (3.18) by $s$

$$\nabla \times \left( \frac{1}{\sqrt{\varepsilon'(r')}} \nabla \times H(r'/s) \right) = \left( \frac{\omega}{cs} \right)^2 H(r'/s)$$

(7.31)

i.e. this is just the master equation again, where mode profile $H'(r') = H(r'/s)$ and $\omega' = \omega/s$. That is, if we want to know the new mode profile after changing the length scale by
a factor $s$, we just scale the odd mode and its frequency by the same factor. The solution of the problem at one length scale determines the solutions at all other length scales.

Just as there is no length scale, there is also no fundamental value of dielectric constant because we assume that the medium has to be macroscopic for the sake of the corresponding Maxwell equations. Here we consider the harmonic mode with a dielectric configuration that differs by a constant factor $s$: $\varepsilon'(r) = \varepsilon(r)/s^2$. Substituting $s^2\varepsilon'(r)$ in the master equation (7.16) yields

\[ \nabla \times \left( \frac{1}{\varepsilon'(r)} \nabla \times H(r) \right) = \left( \frac{s\omega}{c} \right)^2 H(r) \]

(7.32)

In the new system, the harmonic modes remain unaffected, but the frequencies are scaled by the factor $s$. So, if the dielectric constant is multiplied by an overall factor 1/4, the mode patterns do not change although the frequencies double. Therefore, we can use these scaling properties of a PBG material for the polarimeter to be used for wide range of wavelengths.

7.2 Finite-Difference Time-Domain (FDTD) method

The finite-difference time-domain (FDTD) method [54] has been widely used to study EM properties of arbitrary dielectric structures. One use Yee’s discretization [55] scheme to solve Maxwell’s curl equations in real space and time. All field variables are defined on a rectangular grid as shown in figure 7.7. This geometry suggests itself to the line integral versions of Maxwell’s equations (Faraday’s and Ampere’s law), where
electric and magnetic fields are temporally separated by one-half time step. In addition, they are spatially interlaced by half a grid cell.

Based on this scheme, center differences in both space and time are applied to approximate Maxwell’s equations. The expression for the first partial space derivative of $u$ in the x-direction, evaluated at the fixed time $t_n = n\Delta t$ [54]:

$$\frac{\partial u}{\partial x}(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = \frac{u_{i+1/2,j,k}^n - u_{i-1/2,j,k}^n}{\Delta x} + O((\Delta x)^2)$$

(7.33)

Here, we note the $\pm1/2$ increment in the $i$ subscript (x-coordinate) of $u$, denoting a space finite-difference over $\pm(1/2)\Delta x$. Yee chose this notation because he wished to interleave his $E$ and $H$ components in the space lattice at intervals of $\Delta x/2$. And the expression for the first time partial derivative of $u$, evaluated at the fixed space point $(i, j, k)$, follows by analogy [54]:

$$\frac{\partial u}{\partial t}(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = \frac{u_{i,j,k}^{n+1/2} - u_{i,j,k}^n}{\Delta t} + O((\Delta t)^2)$$

(7.34)

Figure 7.7 All field variables defined on a rectangular grid for FDTD simulation
Now the $\pm 1/2$ increments is in the $n$ superscript (time coordinate) of $u$, denoting a time finite-difference over $\pm (1/2)\Delta t$. Yee also chose this notation because he wished to interleave $E$ and $H$ components in time interval of $\Delta t/2$ for purpose of implementing a leapfrog algorithm. Information on mesh size can be found in [56, 57]. A criterion for numerical stability is given in [54] whose physical meaning is a condition for causality.

In this method, one simulates a space of theoretically infinite extent with a finite computational cell due to practical limitations of computer power and memory requires the termination of the computational grid. Any such termination method must not affect the computations inside the finite computational grid, which means the grid termination must not result in reflection of propagating waves back into the grid. To accomplish this, a number of boundary conditions, such as Berenger’s perfectly matched layer (PML) [54] and Liao’s absorbing boundary condition [58], have been proposed that absorb outgoing waves at the computational cell boundaries. Then, an oscillating wave source is introduced into the grid and the entire grid is time-stepped until steady state oscillations are obtained at all points on the grid. Thus, all the field distributions can be stepwise calculated from a start distribution with respect to the above boundary conditions.

For two-dimensional photonic crystal, we use the Yee algorithm with the position

\[
\begin{array}{cccc}
E_y(i-1,j-1) & E_y(i-1,j) & E_y(i-1,j+1) & E_y(i,j-1) \\
H_x(i-1,j-1) & E_y(i-1,j) & E_y(i-1,j+1) & E_y(i,j-1) \\
E_z(i,j) & E_z(i,j) & E_z(i,j+1) & E_z(i,j) \\
H_y(i,j-1) & H_y(i,j) & H_y(i,j+1) & H_y(i,j) \\
H_x(i,j-1) & E_y(i,j) & E_y(i,j+1) & E_y(i,j) \\
E_z(i+1,j-1) & H_y(i,j) & H_y(i,j+1) & H_y(i,j) \\
H_x(i+1,j-1) = E_y(i+1,j) & H_y(i,j) & H_y(i,j+1) & H_y(i,j) \\
\end{array}
\]
of the electric field and magnetic field on 2D grid as shown in Figure 7.8, where $E_z$, $H_x$, $H_y$ components and $H_z$, $E_x$, $E_y$ components are used for TE mode and TM mode simulation, respectively. This algorithm centers its E and H component in time in what is termed a leapfrog arrangement. All of the E computations in the modeled space are completed and stored in memory for a particular time point using previously stored H data. Then all of the H computations in the space completed and stored in memory using the E data just computed. This process continued until time stepping is concluded.

7.3 Fabrication Examples

We have developed a design to realize a PBG cavity in a thin diamond film. The PBG configuration is produced by the presence of a periodic array of holes of the same diameter, (with a larger diameter hole in the center for a cavity mode). The first step is to perform the Finite-Difference Time-Domain (FDTD) simulation to optimize the parameters of this design ($\varepsilon/\varepsilon_0 \sim 2.42$) where the pitch between two holes of the array is 675nm and the diameter of hole is 540nm.

Here, we use thin diamond film, which is called UNCD (Ultananocrystalline Diamond) developed by Argonne National Laboratory with unique properties particularly
suitable to development of novel MEMS/NEMS [59]. Our design parameters such as diameter of holes and the pitch between them are submicron scale and the uniformity of the array is critical issue, so that we use e-beam lithography technique instead of normal photo-lithography. In addition, we chose the lift-off process because the Al pattern may not be able to be created for this structure due to isotropic property of Al wet etching.

The processing step we employed to fabricate a regular array of holes are shown in Figure 7.9. The fabrication started with growth of thin diamond film on silicon substrate (~1um) by MPCVD [59]. Then the UNCD on silicon substrate was cleaned with...
H₂O₂:H₂SO₄ = 1:3 for 15 minutes, and then was dehydrated at 170° C on hot plate for 5 minutes before the lift-off process. We used the Bilayer PMMA 950C5/495C5 to form good step walls on the UNCD for better lift-off process. After spin coating of PMMA 495C5 with 2500 RPM for 1 minutes on UNCD substrate, it was baked at 170° C on hot plate for 5 minutes. Then, second PMMA 950 C5 was spin coated with 4000RPM for 1 minute, followed by baking at 170° C on hot plate for 5 minutes. Then, the desired patterns of holes were generated using a CAD program in an electron-beam lithography system. This pattern was transferred to the PMMA substrate using E-beam lithography. That is, the sample was exposed with E-beam lithography system (Raith) and was developed with MIBK:IPA = 1:3 for 1 minute, followed by soaking a sample in IPA for 30 seconds. Next, Al metal film was deposited by electron-beam evaporation (Varian), where Al is used as a mask material due to its resistance to oxygen reactive ion etching (RIE), which is employed to etch UNCD. Then the sample was soaked in acetone for 30 minutes to remove the remained PMMA and the metal on top of it, which is called “lift-off process”. Finally, the pattern was transferred into the UNCD by reactive ion etching (RIE) with an oxygen plasma (50 mTorr, 30sccm, 200 W) for various time until the exposed UNCD is etched away (the PMMA can also be removed during the RIE etching), followed by removal of Al mask using wet chemical etching.

PMMA patterns on UNCD sample by exposing it with E-beam machine was checked with AFM to see the depth profile (Figure 7.10) and with SEM to analyze the surface profile (Figure 7.11). Since PMMA has refractive index of 1.491, this PMMA pattern itself is a sort of photonic band gap material with air holes in PMMA. As we can see from Figure 7.10 and 7.11, the pitch was 646nm and the diameter of this pattern was
560nm, which was little bit different from the expected values due to some process parameters.

After making the PMMA pattern on UNCD film, we create the metal pattern as shown in Figure 7.11, where regular array of air holes in metal is made. This pattern is

![AFM image for PMMA pattern on thin diamond film](image.png)

**Figure 7.10** AFM image for PMMA pattern on thin diamond film
another type of PBG structure with almost the same geometry as previous PMMA pattern and different ratio of dielectric constant \( (\varepsilon/\varepsilon_0: 8 \sim 10) \). Currently, we are developing the RIE etching process to make etched holes in this diamond film, but the finding the RIE processing conditions are challenging task because of both the size of hole \((\sim\) submicron length\) and the hardness of the diamond film. Thus, we need to develop and analyze the RIE etching condition to create the regular array of air holes in a thin diamond film.

**Figure 7.11** SEM image for PMMA pattern on thin diamond film

**Figure 7.12** An example of a two-dimensional photonic crystal: a square lattice of Al rods with radius \( r \sim 260\text{nm} \), relative dielectric constant of 8~10
Chapter 8

Volume-Grating in a Photonic Band Gap Material

An inherent polarization sensitivity of a volume grating, in general, can be used to determine all the Stokes parameters of input beams. Here we show simulation results for a PBG, a point defect mode and a line defect mode for photonic crystals, and then demonstrate the polarization-dependent photonic band gap using Finite-Difference Time-Domain method. In the last section, we propose a volume-grating Stokesmeter using the polarization-dependent photonic band gap, and derive the basic principle of this device.

8.1 Simulation Results for a PBG, a Cavity Mode, and a Waveguide Mode

We apply simple schemes based on FDTD methods to calculate both the photonic band gaps and the associated field distribution of perfect crystals. Depending on the purpose of the simulation, either absorbing or periodic boundary conditions are applied. A schematic of the computational cell is shown in figure 8.1. A slab of photonic crystal is

![Figure 8.1 Schematics of the computational cell used in the transmission calculation](image)
placed in the middle of the cell with its left and right surfaces normal to the y direction. Plane waves propagating along the y axis are generated by exciting plane waves. On the other side of the crystal, the field amplitude, is monitored at a single point, marked “detector”. Liao’s absorbing boundary conditions [57] are used at the left and right edges of the computational cell. Plane waves hitting these boundaries get absorbed. On all other boundaries (top and bottom edges), we use periodic boundary conditions. By placing one unit cell of a slab of photonic crystal in the computational cell, we can

![Figure 8.2 Simulation results on plane waves normally incident on a 2D slab composed of six rows of 4mm diameter Pyrex rod on a 9mm square lattice](image)

*Figure 8.2 Simulation results on plane waves normally incident on a 2D slab composed of six rows of 4mm diameter Pyrex rod on a 9mm square lattice*
simulate plane waves normally incident upon a slab with infinite extent in the $x$ and $y$ directions. Figure 8.2 shows the simulation results on a PBG material composed of six rows of 4mm diameter Pyrex rods ($\varepsilon_r = 4.2$) on a 9mm square lattice. While the plane wave with frequency at 10 GHz is completely transmitted through the PBG structure with no reflection, the plane wave with frequency of 14 GHz in the band gap is completely reflected from the PBG structure, resulting in almost zero transmission. Instead of studying the steady-state response, one frequency at a time, we choose to send a single pulse of light with a wide frequency profile. The incident amplitude is calibrated at the detector point by running a simulation without the crystal. Simulations are then performed with the crystal present. Here, the amplitude at the detector describing the transmitted wave. The transmitted and incident amplitudes are then transformed into the frequency domain using Fast Fourier Transformations. The transmission coefficients are then determined by taking the square of the ratio between the two amplitudes. Figure 8.3
Figure 8.3 The magnitude of the transmission coefficient for the periodic array of Pyrex rods in air shows the magnitude of the transmission coefficient for the periodic array of Pyrex rods in air in frequency range of 1GHz to 25 GHz. The simulation result shows that photonic band gaps are present between 12 GHz and 15 GHz for this 2D photonic crystal.

A cavity mode, where light can be localized or trapped in the defect, can be created by making a point defect in a photonic crystal. The frequency, symmetry, and other properties of the defect mode can be easily tuned to anything desired. We show a defect mode analysis using the FDTD approach for the 2D square lattice of alumina rods ($\varepsilon/\varepsilon_o = 8.9$) in air. Here, we used 5 X 5 supercell with the center rod removed ($r = 80\text{nm}$, $a = 400\text{nm}$), and 200 grids are selected along each direction. We create the sinusoidal source field in the center of the array with frequency of 295 THz, resulting in light
confinement at this frequency as shown in Figure 8.4. Here, the color indicates the magnitude of the electric field oriented in z-direction.

In addition, we have created a waveguide mode by removing the rods in the center column line and then have performed the FDTD simulation for this structure. Figure 8.5 shows a electric field (Ez) distribution of this channel mode in a 2D square lattice with rods ($\varepsilon/\varepsilon_0=11.6$) in air using 11 X 11 supercell, where the radius of hole is 150nm and the pitch between the consequent holes is 400nm. As we can see from the simulation result, the light wave at frequency of $10^{15}$ Hz is confined along the missed line, resulting in a waveguide mode.

*Figure 8.4 Electric field (Ez) distribution of a defect mode in a 2D square lattice with aluminum rod in air*
It has recently been shown that the polarization of electromagnetic radiation can be completely controlled using devices constructed entirely from photonic crystals. This polarization-dependent band gap characteristic has been experimentally observed for propagation in the plane of periodicity in both slab and bulk two-dimensional (2D) photonic crystals [59-61], and have been used to create novel waveplates and polarizing beamsplitters [62].

Polarization-dependent effects in photonic crystals can be qualitatively understood by considering the response at each material interface for different polarizations. Clearly, a one dimensional photonic crystal has a polarization-independent...
response to normally incident EM radiation because each polarization ‘sees’ the same boundary conditions at each material interface as it propagates through the crystal. However, if the waves are not normally incident; a basic result of EM theory shows that so called s- and p-polarizations have different Fresnel coefficients for waves that are obliquely incident on a dielectric interface. Thus, we expect that a one-dimensional stack in general produce a polarization-dependent band structure for obliquely incident waves. The situation becomes considerably more interesting when we consider photonic crystals with two- or three- dimensional periodicity. The polarization properties associated with the band structure of a 2D crystal can be understood considering the anisotropy of the lattice. Each polarization experiences different boundary conditions depending on whether it is parallel or perpendicular to the plane of symmetry, resulting in different Fresnel coefficients for s- and p-polarizations Thus, the band structures for TM and TE waves are typically different for two-dimensional crystals, with the band gap centre frequency, width, depth and shape all depending on polarization [63,64]. The Bragg reflections from many such interfaces will lead to polarization-dependent properties for a 2D photonic crystal.

Photonic-crystal based devices for polarization control are particularly important because the polarization-dependent properties of PBG materials can be tailored by adjusting the lattice geometry and distribution of materials within the unit cell, size and dimensionality, as well as the refractive indices of the constituent materials. Thus, the properties of photonic crystals can be engineered to suit a wide variety of diverse applications depending on the boundary conditions. In this respect, these devices are far more versatile than conventional devices for polarization control. In addition, since
Maxwell’s equations are scale-invariant, results obtained at one portion of the EM spectrum apply equally well across the entire spectrum with appropriate scaling of the structure, as long as materials of comparable permittivity and permeability can be found at the new wavelength. Thus, if we can implement a volume-grating polarimeter using these two properties of photonic crystals, then this device can be operated in a wide frequency range, and used for a high intensity input beam.

Here, we demonstrate this polarization–dependent photonic band gap using the FDTD simulation. The PBG consists of a square lattice of 6 by 4mm diameter alumina rod as we used in the previous section. The periodicity of the array is 9mm along the two axes as illustrated in Figure 8.1. We can simulate a light wave function normally incident upon a slab with infinite extent in the $x$ and $y$ directions by placing one unit cell of a slab of photonic crystal in the computational cell as shown in Figure 8.6.

![Figure 8.6 schematic diagram for PBG calculation](https://example.com/figure86.png)


The PBG structure used here is shown in Figure 8.7 with the directions of the plane waves for TE and TM mode as a source. From this modeled space, we have performed the FDTD simulation for both TE and TM modes by sending a single pulse of light with a wide frequency profile multiplied with a sinusoidal wave function. The output amplitudes are calibrated at a point after the PBG structure by running a simulation both without and
with the crystal. After performing a Discrete Fourier Transform for the transmitted and incident amplitudes, we can get the transmission spectrum as shown in Figure 8.8. While the deep absorptions in the frequency range of 12 GHz ~ 15 GHz and 23 GHz ~ 26 GHz are produced for TE mode case, the shallow absorption around 17 GHz and the deep absorption around the frequency of 27 GHz are observed. Note that the direction of electric field component for the TM mode is in parallel with the plane of the photonic crystal array, resulting in attenuation of the transmitted beam near DC component.

Here, we chose the TE mode to demonstrate the polarization dependent property of this photonic crystal, so that we can easily produce arbitrary polarization states of the input beam by rotating the direction of the electric field in yz-plane. That is, $s$-
(p-polarization) state corresponds to the direction of z-axis (y-axis), and any linear polarization state in yz-plane can be produced by an angle $\theta$ as shown in Figure 8.7.

The transmission spectrum of this PBG structure clearly has a dependence on the polarization state of the input beam as illustrated in figure 8.9. We found that as the angle of polarization of the incident wave in yz-plane was varied from 0 degree to 90 degree, the absorption spectrum increase, resulting in a decrease of the transmitted beam. For example, the relative ratio of the absorptions for two different frequencies (e.g. 14 GHz and 15 GHz) in the photonic band gap shows a sinusoidal dependence as shown in Figure 8.10. This is very similar to the polarization dependent diffraction efficiency from the volume holographic grating [29].
In this section, we propose a volume-grating Stokesmeter using a polarization-dependent photonic band gap (PBG) structure as shown in Figure 8.11, which consists of a pair of spatially separated volume gratings in a photonic crystal with two electro-optic
modulators. The resulting measurement matrix relates the observed signals to the incident Stokes vector will have the following general relation:

\[
\begin{bmatrix}
I_{t1} \\
I_{t2} \\
I_{t3} \\
I_{t4}
\end{bmatrix} =
\begin{bmatrix}
A_1 + B_1 & (A_1 - B_1)\cos(2\gamma_1) & (A_1 - B_1)\sin(2\gamma_1) & 0 \\
A_2 + B_2 & (A_2 - B_2)\cos(2\gamma_2) & (A_2 - B_2)\sin(2\gamma_2) & 0 \\
A_1 + B_1 & (A_1 - B_1)\cos(2\gamma_1) & 0 & -(A_1 - B_1)\sin(2\gamma_1) \\
A_2 + B_2 & (A_2 - B_2)\cos(2\gamma_2) & 0 & -(A_2 - B_2)\sin(2\gamma_2)
\end{bmatrix}
\begin{bmatrix}
I \\
Q \\
U \\
V
\end{bmatrix}
\]

This relation is similar to the case of the HSM [30]. Here, \((I, Q, U, V)\) represents the input Stokes parameters to be determined. These values are found by making measurements of the transmitted beams during two time periods as shown at the bottom of figure 1. During the interval \(T_1\), \(I_{t1}\) \((I_{t2})\) are found from the transmitted beam intensity from the 1st \(2^{nd}\) volume-grating in a photonic crystal. \(I_{t3}\) \((I_{t4})\) is the transmitted beam intensity from the 1st \(2^{nd}\) volume-grating during the interval \(T_2\). The electro-optic modulators, active during

![Diagram of the photonic crystal based polarimetric imaging system](image)

Figure 8.11 Proposed architecture for a photonic crystal based polarimetric imaging system

During the interval \(T_2\), serve to interchange the U and V parameters by rotating the fast axis of the second EOM by 45º with respect to that of the first EOM. \(A_1, A_2\) \((B_1, B_2)\) characterize the transmitted beam intensity from the two gratings for s (p)-polarized input beams. The
angle $\gamma_1 (\gamma_2)$ denotes the rotation of the substrate containing grating 1 (2) in order to have robust measurement matrix. By measuring all four transmitted beam intensities for a specific frequency in a photonic band gap during interval $T_1$ and $T_2$, we can find the input Stokes parameters in a similar way as we did in the HSM.

Since the architecture requires two different contrast ratios between the p- and s-polarization components, volume-grating elements in this system can be a pair of spatially separated PBGs with two different numbers of rows of holes because the level of the attenuation of the transmittance will vary depending on the number of grids. Or these elements can be constructed with a pair of spatially separated PBG structure, where the diameter of holes are different, but the length between the two holes are same order to use the fact that a photonic band gap in square-type 2D structure will be shifted as we change the diameter of the regular array of holes. In either way, we can create the polarization-dependent transmission spectrum for a specific frequency with two different contrast ratios of the transmitted beam intensities for p- and s- polarization case.

It is also possible to implement the spectrally resolved volume-grating Stokesmeter using a photonic crystal by replacing the original PBG with other one, which has different length scale on the diameter and the pitch of the array or different ratios of the dielectric constants, resulting in the variation of the transmitted beams over a different band gap based on the scaling property of a PBG structure. Furthermore, some researcher have developed many structures activated by several external parameters, including electric field (E field) [65,66], temperature [67], magnetic field (H field) [68], and strain [69] to implement tunable band stop of a photonic crystals. This tunability of band gaps is usually achieved by controlling the electric permittivity or the magnetic permeability of
the constituent materials as the Bragg gap positions are governed by the average refractive index of the composite. Thus, if we use some of these methods to realized the tunable PBG structure in our architecture, then we will be able to implement PBG based spectrally resolved polarimeters.
Chapter 9

Conclusion

9.1 Summary

We have developed a polarization imaging architecture using thick multiplexed holograms that has many advantages over current polarimetric imaging techniques. The analysis showing the principle of operation explicitly takes into account the polarization state of the incident and diffracted wave fields. Transformation of initial Stokes parameters by grating diffraction is formulated by a Mueller matrix defined in terms of diffracted amplitudes of planar and perpendicular polarization components. A procedure is outlined using two sets of rotated orthogonal gratings and a quarter-wave plate to compute all four unknown Stokes parameters required for polarimetric imaging. Highly polarization-sensitive holographic gratings required for a holographic Stokesmeter has been made. These gratings can be accurately described by coupled-wave analysis. A numerical analysis of the noise tolerance of the Stokesmeter has been performed by considering the effect of additive white Gaussian noise (AWGN), and the gratings demonstrated allow the construction of a holographic Stokesmeter. The use of a heterodyne receiver architecture can lead to additional gains in the signal-to-noise ratio.

We also have performed measurements of arbitrary Stokes parameters required for polarimetric imaging with the HSM that consists of two sets of rotated orthogonal gratings and a quarter-wave plate. Measured Stokes parameters are compared with the ones obtained by a conventional method. The values measured by both methods are
approximately equal to the assumed ones within the measurement error range. This shows the feasibility of an HSM in its original configuration. A unique feature of the HSM is that it can be designed for spectral multiplexing at a high speed by replacing the single hologram in each zone of the substrate with a set of angle multiplexed gratings. We have demonstrated this spectrally scanned HSM by measuring arbitrary Stokes parameters of input beams for two different wavelengths. The ability to combine spectral discrimination with polarization imaging in a single device makes this HSM a unique device of significant interest.

Photonic Band Gap materials with spatially periodic dielectric functions are a particular class of holograms with extremely high refractive index contrast. The refractive function of volume gratings in a material can be approximated by a small number of plane waves, resulting in the photonic crystal periodicity. We briefly describe general properties of photonic crystals and show some fabrication examples to implement two-dimensional PBG structures. We have performed the Finite-Difference Time-Domain (FDTD) simulations to show photonic band gaps for a regular array of holes in a photonic crystal as well as to demonstrate defect modes by creating a point defect and a line defect in this structure. We have further demonstrated a polarization-dependent photonic band gap by analyzing the transmission of a normally incident waves through the 2D PBG structure based on the FDTD approach. Using this polarization-dependent band gap property, we construct a new type of volume-grating Stokesmeter that consists of one set of two different phonic band gap structures. In addition, we describe a spectrally resolved polarimeter using the scaling property of a photonic band gap material.
9.2 Future work

The HSM we described in this thesis needs some improvements before it can become a realistic imaging system. The reference optical elements we used contribute to the error observed in the HSM data. Thus, we need to use more precise optical elements to suppress these errors. In addition, the gratings we used for the experiment had contrast ratio in the range of 40 ~ 50%. Since residual errors in the HSM drops monotonically with increasing contrast ratio as we described in chapter 5, we expect that a HSM made with higher contrast ratio will produce more accurate results. Thus, in order to get better polarimetric imaging data we should use precise optical elements as well as high-contrast holographic gratings.

The new volume-grating Stokemeter we proposed in chapter 8 needs to fabricate a PBG material with a good uniformity of regular array holes using conventional micro- or submicro- fabrication techniques depending on the frequency range of the input beams. Thus, we need to setup process conditions such as metal deposition, photo or E-beam lithography, chemical wet/ dry etching to implement a PBG structure. We are currently in the development of 2D PBG structure in a thin diamond film, which requires precise controls of Lift-Off process as well as E-beam lithography. This PBG structure can be used to implement the volume-grating Stokesmeter as well as to create a high Q cavity mode and a channel wave guide mode in the visible range.

Moreover, we may need to develop a FPGA processor suitable for these volume-grating Stokesmeters to take advantage of the high speed operation.
List of publication

Journal Articles


Conference Presentations


Reference


